Cooperative Moving-Target Enclosing of Networked Vehicles With Constant Linear Velocities

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***Abstract*—This paper investigates the cooperative moving-target enclosing control problem of networked unicycle-type nonholonomic vehicles with constant linear velocities. The information of the target is only known to some of the vehicles, and the topology of the vehicle network is described by a directed graph. A dynamic control law is proposed to steer the vehicles, such that they can get close to orbiting around the target while the target is moving with a time-varying velocity. Besides, the constraint of bounded angular velocity for the vehicles can always be satisfied. The proposed control law is distributed in the sense that each vehicle only uses its own information and the information of its neighbors in the network. Finally, simulation results of an example validate the effectiveness of the proposed control law.**

***Index Terms*—Cooperative control, nonholonomic vehicles, target enclosing, velocity constraint.**

I. Introduction

RECENT years have witnessed the rapid development in control of multi-agent systems, see [1–8] and references therein. In particular, numerous research efforts have been made to cooperative control of multiple unicycle-type nonholonomic vehicles due to its wide potential applications, such as sensor networks, environment exploration, robotic surveillance and entertainment. On the one hand, the unicycle model can be used to describe not only a mobile wheeled robot (MWR) [9–12], but also other robotic systems, such as the simplified model of an unmanned aerial vehicle (UAV) equipped with standard autopilots or a fixed-wing UAV [11, 13]. On the other hand, the group motion of multiple vehicles for a common objective can cooperatively complete the task, lower the cost and improve the efficiency. As reviewed in the recent survey [14], observing moving targets is an important application of multi-vehicle systems, and much research attention has been paid to control techniques for monitoring moving targets by cooperative vehicles.

Enclosing the target by a group of vehicles is one typical and effective method for monitoring, for example, the ocean sampling [15]. The target-enclosing control problem of unicycle-type vehicle is a fundamental problem and has attracted much attention since the last decade. Starting from the cyclic pursuit problem of vehicles, it was assumed in most existing works that the linear velocities of vehicles are constant. In [16], the formation of multiple vehicles with identical constant linear velocities was considered and the stability analysis on the linearized system was given. Later, the periodic formations was considered in [17]. In [18, 19], the cyclic pursuit problem of vehicles with nonidentical constant linear velocities and nonidentical desired radii was studied. For a stationary target which was also called beacon or center, it was assumed in most existing works that the information of the target is available to each vehicle. In [20, 21], a comprehensive investigation on vehicles with identical linear velocity was presented. A gradient control law based on potential function was developed in [20] for networked vehicles with a complete graph condition which was extended to a balanced graph condition in [21]. Later in [22], the problem of enclosing a target with nonidentical radii by vehicles with nonidentical linear velocity was considered. In [23], the sensory limitation in visibility of each vehicle was taken into account, and equilibrium configurations of the multi-vehicle systems were analysed. The result was also validated with experiments in [24]. In [25], a hybrid switching control law based on the relative distance between each vehicle and the target was proposed, and it was shown that vehicles are able to move directly to the target when they are far away from the target. In [26], a novel distributed solution was proposed such that the control laws are heterogeneous for vehicles but different orbits centered at a common target can be achieved. In [27], it was shown that the proposed controller only requires each vehicle to use bearing angle measurements. Similar result was obtained in [28] and the size of a disk-like target was additionally taken into account. Some existing works further considered the case where the information of the target is only known to some of the vehicles. In [29], a range-defined jointly connected proximity graph condition was used for networked vehicles, and the target enclosing can be achieved if there is a specified vehicle orbiting around the target. In [30], networked vehicles with a cycle graph condition was studied. In [31], the case for networked vehicles with a cycle graph condition and the problem of enclosing a target with nonidentical radii was investigated. However, these approaches were developed for the cyclic pursuit problem or the stationary-target enclosing

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control problem, which cannot be applied to moving-target enclosing control problem.

The main challenge in controller design for moving-target enclosing is that each vehicle needs to orbit around the target, and simultaneously the center of its circular motion needs to track the trajectory of the target. For unicycle-type vehicles, some existing works focused on the case where the velocity of the moving target is constant. In [32, 33], it was shown that enclosing the moving target can be achieved when the linear velocity of the vehicle maintains constant and the target is also a unicycle-type vehicle. In [34–36], several approaches based on Lyapunov guidance vector field were developed, and adaptive control laws were proposed for the case where the target moves with an unknown constant velocity. As pointed out in [37], the moving-target enclosing control problem becomes more difficult if the target moves with a time-varying velocity, even though it is known to each vehicle. In [37], a distributed control law was proposed and the tracking errors with respect to circular motions around the moving-target were locally uniformly bounded. In order to make the tracking errors converge to zero, a translation control design using the inter-vehicle communication and the measurements of vehicles and the target in inertial frame was proposed in [38]. In [39], a distributed controller was developed with only requiring each vehicle to use measurements in its local coordinate frame. In [40], two control strategies based on backstepping techniques were developed in Cartesian coordinates and polar coordinates respectively. Recently in [41], the assumptions in [38] was relaxed at the price of making the tracking errors converge to an arbitrarily small neighborhood of zero. In [42], an approach to moving-target enclosing control of vehicles with limited field of view was presented.

However, all aforementioned works on the moving-target enclosing control problem requires the assumption that the information of the target is known to all vehicles, including the (relative) position, velocity, and/or acceleration of the target. This assumption inevitably limits the scope of applications. For example, in a large group of vehicles, there may exist only one or some of the vehicles equipped with the advanced sensors, and only these vehicles are able to measure the states of the target. This practical scenario motivates our study. In fact, several works have investigated the moving-target enclosing control problem for vehicles with simple dynamics, such as single- or double-integrators. Most of these results also required all vehicles to obtain some information of the target, for instance [43–46]. Concerning the case where some of the vehicles do not know the target, these results are extendable by adopting some cooperative control approaches for multi-agent systems, for example [47–49]. Recently, the problem of enclosing multiple moving targets by double-integrators was studied in [50, 51] where some vehicles exchange the information with other vehicles. However, for unicycle-type vehicles, the cooperative moving-target enclosing control problem becomes more complicated and the stability analysis turns more difficult. Existing approaches for multi-vehicle systems with simple dynamics cannot be directly extended.

In this paper, the network topology among vehicles and the target is described by a directed graph. It is shown that under a dynamic control law, vehicles with nonidentical constant linear velocities are able to get close to traveling along heterogenous circles centered at the target as it moves with bounded time-varying velocity. Moreover, the constraint of bounded angular velocity for each vehicle can always be satisfied.

The contribution of this paper is summarized as follows. First, to the best of our knowledge, all existing results on the moving-target enclosing control of unicycle-type vehicles required the centralized assumption that all vehicles have the access to the information of the target, for instance, [32–41]. The main contribution of this paper lies in relaxing this centralized assumption. The proposed control law is distributed in the sense that each vehicle uses its own information and information of its neighbors. This paper can be viewed as the first attempt to solve the cooperative target enclosing control problem of unicycle-type vehicles under the assumption that only some of the vehicles (at least one) know the information of the target. Second, the proposed control law enables vehicles get close to orbiting around the moving target as in [37, 41]. It is not only shown that the tracking error is globally uniformly ultimately bounded rather than locally uniformly bounded in [37], but also proved that the trajectory of each vehicle can converge to a moving-target enclosing motion with first-order smallness. Note that the price of relaxing the aforementioned centralized assumption is an ultimate bound of the tracking error. Finally, compared with [32–35, 37–41], the assumption that the desired radii around the target are not identical, which extends the scope of practical application. Compared with [37–41], our result can be applied to the scenarios where vehicles are required to cruise at constant linear velocities. For instance, the constraint of constant linear velocity applies with some low-cost UAVs, e.g., Aerosonde [52]. Moreover, compared with those existing controllers, our design ensures the constraint of bounded angular velocity to be always satisfied.

The remainder of this paper is organized as follows. In Section II, we introduce the problem setting and give the problem formulation. In Section III, we first propose a dynamic control law and then present the stability analysis on the resulting closed-loop system. In Section IV, simulation results are shown to illustrate the main result. In Section V, the conclusion is drawn.

Notations: The following notations are used throughout the paper. For a vector $\boldsymbol{x} \in \mathbb{R}^n$ and a matrix $\mathcal{K} \in \mathbb{R}^{n\times n}$, $\|\boldsymbol{x}\| = \|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$, $\|\boldsymbol{x}\|_{\infty,T} = \sup_{t\geq T}\|\boldsymbol{x}(t)\|_\infty = \sup_{t\geq T}(\max_{i=1}^{n}|x_i(t)|)$, and $\|\mathcal{K}\| = \|\mathcal{K}\|_2$. Function $\mathrm{sign}(x) : \mathbb{R} \to \{1, -1\}$ is the signum function, and $\otimes$ denotes the Kronecker product on two matrices.

II. Problem Formulation

Consider a group of N unicycle-type nonholonomic vehicles, and the kinematics of each vehicle i, $i = 1, 2, ..., N$, is described by:

$$\dot{x}_i = v_i \cos\theta_i,\ \ \dot{y}_i = v_i \sin\theta_i,\ \ \dot{\theta}_i = \omega_i \tag{1}$$

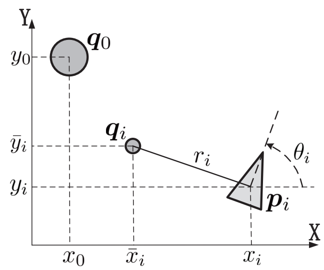

Fig. 1. Illustration of the variables.

where $\boldsymbol{p}_i := [x_i \ y_i]^{\mathrm{T}} \in \mathbb{R}^2$ and $\theta_i \in \mathbb{R}$ are the position and the heading angle respectively; see Fig. 1 for an illustration. $v_i \in \mathbb{R}$ and $\omega_i \in \mathbb{R}$ are linear velocity and angular velocity respectively. As in [16, 17, 20, 21, 23, 32, 33], v_i is assumed to be a positive constant, and ω_i is used as the control input of system (1). Different from [16, 17, 20, 21], the constant linear velocities are allowed to be nonidentical. Moreover, the following constraint of angular velocity is considered:

$$-\omega_i^{\max} \leq \omega_i \leq \omega_i^{\max} \tag{2}$$

where $\omega_i^{\max}$ is a nonidentical given positive constant of vehicle i as the bound of angular velocity.

The objective is to steer each vehicle such that it can orbit around a target with a given radius $r_i > \frac{v_i}{\omega_i^{\max}}$. The target is assumed to move in a trajectory $\boldsymbol{q}_0 := [x_0 \ y_0]^{\mathrm{T}}$ with bounded velocity and acceleration; see Fig. 1 for an illustration. That is, $\boldsymbol{v}_0 := \dot{\boldsymbol{q}}_0$ satisfies the following assumption.

Assumption 1: The velocity of the target $\boldsymbol{v}_0(t)$ is bounded for all $t \geq t_0$, and $\dot{\boldsymbol{v}}_0$ exists and is also bounded. ■

In this paper, we consider the case where only some of the vehicles know the information of the target. The multi-vehicle systems are required to be installed with the sensors and communication devices. The vehicle network is built based on the setting where each vehicle only has access to the information of its neighbors in a network. The concerned network is described by a directed graph $\bar{\mathcal{G}}$ as follows.

First, the network among N vehicles is described by a directed graph $\mathcal{G} = \{\mathcal{O}, \mathcal{E}\}$, where $\mathcal{O} = \{1, ..., N\}$ is a finite set of nodes representing N vehicles, and $\mathcal{E} \subseteq \{(j,i) : j \neq i, i, j \in \mathcal{O}\}$ is a set of edges containing directed edges from node j to node i. Denote the target by node 0. Then, a directed graph $\bar{\mathcal{G}} = \{\bar{\mathcal{O}}, \bar{\mathcal{E}}\}$ with $\bar{\mathcal{O}} = \mathcal{O} \cup \{0\}$ and $\bar{\mathcal{E}} = \mathcal{E} \cup \{(0,i), \ i \in \mathcal{O}\}$, can be obtained by combining $\mathcal{G}$ and node 0. Vehicle i has access to the information of vehicle j if $(j,i) \in \mathcal{E}$, $j \neq i$, or the target if $(0,i) \in \bar{\mathcal{E}}$. Node j is called the neighbor of node i if $(j,i) \in \bar{\mathcal{E}}$, and all neighbors of node i locates in a set $\bar{\mathcal{N}}_i \subseteq \mathcal{O}$. Define $a_{ij} = 1$ if $(j,i) \in \bar{\mathcal{E}}$, otherwise $a_{ij} = 0$. Finally, graph $\bar{\mathcal{G}}$ is assumed to satisfy:

Assumption 2: The directed graph $\bar{\mathcal{G}}$ contains a directed spanning tree with node 0 being the root. ■

According to the objective, the desired group motion can be summarized as a *moving-target enclosing motion* which is formally defined as follows.

Definition 1: A trajectory $\check{\boldsymbol{p}}(t) := [\check{\boldsymbol{p}}_1^{\mathrm{T}}(t)...\check{\boldsymbol{p}}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$ for the group of nonholonomic vehicles (1) is called a *moving-target enclosing motion* if

$$\check{\boldsymbol{p}}_i(t) - \boldsymbol{q}_0(t) = r_i \angle \varphi_i(t), \ \dot{\varphi}_i > 0 \tag{3}$$

for all $t \geq t_0$, where $\boldsymbol{q}_0(t)$ is the position of the moving-target, and r_i is the given radius of vehicle i. ■

Then, the formal problem statement of the *cooperative moving-target enclosing control problem* to be solved in this paper is given as follows.

Problem 1: For each vehicle (1) with any initial state $[\boldsymbol{p}_i^{\mathrm{T}}(t_0) \ \theta_i(t_0)]^{\mathrm{T}} \in \mathbb{R}^3$, $\forall t_0 \geq 0$, given a target $\boldsymbol{q}_0(t)$ moving with velocity $\boldsymbol{v}_0(t)$, set any radius $r_i > \frac{v_i}{\omega_i^{\max}}$, and find a dynamic control law in the form of

$$\begin{aligned} \omega_i &= \varsigma_i(\boldsymbol{p}_i, \theta_i, \boldsymbol{\rho}_i, r_i, v_i) \\ \dot{\boldsymbol{\rho}}_i &= \boldsymbol{\varrho}_i(\boldsymbol{p}_i, \theta_i, \boldsymbol{\rho}_i, \boldsymbol{\rho}_j, r_i, v_i), \ j \in \bar{\mathcal{N}}_i \end{aligned} \tag{4}$$

such that the trajectory of $\boldsymbol{p} := [\boldsymbol{p}_1^{\mathrm{T}}, ..., \boldsymbol{p}_N^{\mathrm{T}}]^{\mathrm{T}}$ converges to a *moving-target enclosing motion* $\check{\boldsymbol{p}}$ defined in Definition 1, i.e.,

$$\lim_{t \to \infty} (\boldsymbol{p}(t) - \check{\boldsymbol{p}}(t)) = \boldsymbol{0} \tag{5}$$

where $\boldsymbol{\rho}_i$ is an designed adaptive state used to estimate the state of the target, and $\varsigma_i(\cdot)$ and $\boldsymbol{\varrho}_i(\cdot)$ are sufficiently smooth functions. ■

Remark 1: Different from [37–41], we studied vehicles with constant linear velocities. Since only one control input channel is considered in this paper, the objective for position or phase distribution around the moving target is not included. As in [32–34, 36], we focus on controller design for enclosing the moving target. It is of practical meaning to consider vehicles with constant linear velocities. Though most existing vehicles have ability to control linear velocities, yet many ground, aerial, or surface vehicles are equipped with the cruise control system or similar servo systems which are used to maintain the steady speed. Some vehicles is constrained to cruise with a constant linear velocity, for example a low-cost UAV called Aerosonde [52]. A vehicle cruising at a constant speed can maintain the operation efficiency, increase the fuel economy, improve the passengers' comfort, and reduce the wear of onboard devices. Due to the advantages of using constant linear velocity, many existing works on target enclosing control of unicycle-type vehicles assumed constant linear velocities [15, 20–23, 32, 33]. In particular, an application to adaptive ocean sampling by a fleet of self-directed underwater gliders was demonstrated in [15]. The strategy used in [15] took into account that gliders effectively operate at constant speed, i.e., constant v_i. ■

Remark 2: Under Assumption 2, only some of the vehicles (at least one) is required to know the information of the target. Assumption 2 is more relaxed than that in [32–41] where all vehicles needs to know some information of the target. Besides, Assumption 2 indicates that the network topology among vehicles is not limited to a cycle [16–19, 37–39]. It is noted that the constant linear velocities and the desired radii are allowed to be nonidentical. This assumption is more relaxed than used in [34, 35, 37–41], where the constant linear velocities and the desired radii have to be identical. Enclosing a target with different radii was also considered in some existing works, such as [22, 26], which can be applied to some practical scenarios. For instance, a fleet of UAVs cruising in different altitudes aim to monitor a target of which the shape is a pyramid in 3-D space. If the UAVs need to monitor the surface of the target at the same distance, they have to choose nonidentical radii relative to the axle wire of the target. ■

III. Main Result

In this section, we first propose a solution to the *cooperative moving-target enclosing control problem*, i.e., Problem 1, and then present the stability analysis on the closed-loop system consisting of the proposed control law and the vehicle system.

A. Control Law Design

First, introduce a coordinate transformation $\boldsymbol{p}_i \mapsto \boldsymbol{q}_i := [\bar{x}_i\ \bar{y}_i]^{\mathrm{T}}$, $i = 1, 2, ..., N$, as

$$\bar{x}_i = x_i - r_i \sin\theta_i,\ \ \bar{y}_i = y_i + r_i \cos\theta_i, \tag{6}$$

and $\boldsymbol{q}_i$ represents the circular motion center of vehicle i; see Fig. 1 for an illustration. Then, system (1) can be transformed into the following system

$$\begin{aligned}\dot{\boldsymbol{q}}_i &= [\cos\theta_i\ \ \sin\theta_i]^{\mathrm{T}}(v_i - \omega_i r_i)\\ \dot{\theta}_i &= \omega_i,\ \ i = 1, 2, ..., N.\end{aligned} \tag{7}$$

System (7) can be viewed as the dynamics of the circular motion center. Thus, to achieve (5), it suffices to show that

$$\lim_{t\to\infty}(\boldsymbol{q}_i(t) - \boldsymbol{q}_0(t)) = \boldsymbol{0},\ \ i = 1, 2, ..., N. \tag{8}$$

However, under Assumption 2, some vehicles do not have access to the target. To handle this situation, we introduce a new variable $\boldsymbol{\eta}_i \in \mathbb{R}^2$ for each vehicle i. $\boldsymbol{\eta}_i$ and $\dot{\boldsymbol{\eta}}_i$ serve as the estimates for the position and velocity of the target respectively. For each vehicle i, the initial states $\boldsymbol{\eta}_i(t_0)$ and $\dot{\boldsymbol{\eta}}_i(t_0)$ can be arbitrary selected in $\mathbb{R}^2$. Besides, set $\boldsymbol{\eta}_0 = \boldsymbol{q}_0$ and $\dot{\boldsymbol{\eta}}_0 = \dot{\boldsymbol{q}}_0 = \boldsymbol{v}_0$ by default.

Then, we propose the following dynamic control law:

$$\omega_i = \frac{1}{r_i}(v_i + [\cos\theta_i\ \ \sin\theta_i]\boldsymbol{\sigma}_i(\boldsymbol{k}(\varepsilon_i)(\boldsymbol{q}_i - \boldsymbol{\eta}_i) - \dot{\boldsymbol{\eta}}_i)) \tag{9}$$

$$\dot{\boldsymbol{\eta}}_i = \frac{1}{c_i}\sum_{j\in\bar{\mathcal{N}}_i} a_{ij}\dot{\boldsymbol{\eta}}_j + \frac{\mu}{c_i}\sum_{j\in\bar{\mathcal{N}}_i} a_{ij}(\boldsymbol{\eta}_j - \boldsymbol{\eta}_i),\ \ j \in \bar{\mathcal{N}}_i \tag{10}$$

where $\mu > 0$ is any constant, $c_i = \sum_{j\in\bar{\mathcal{N}}_i} a_{ij}$, $\boldsymbol{k}(\varepsilon_i) \in \mathbb{R}^{2\times 2}$ is a gain matrix defined as $\boldsymbol{k}(\varepsilon_i) = \mathrm{diag}\{\varepsilon_i, \varepsilon_i\}$ with any constant $\varepsilon_i > 0$, and function $\boldsymbol{\sigma}_i(\cdot)$ is defined as $\boldsymbol{\sigma}_i([z_1\ z_2]^{\mathrm{T}}) = [\mathrm{sat}(z_1)\ \ \mathrm{sat}(z_2)]^{\mathrm{T}}$ with $\mathrm{sat}(z_l) = \mathrm{sign}(z_l)\min(z_l, m_i)$, $l = 1, 2$, and a constant m_i satisfying

$$0 < m_i \le \frac{1}{\sqrt{2}}(\omega_i^{\max} r_i - v_i). \tag{11}$$

The dynamic control law (9)–(10) is in the form of (4), and is distributed in the sense that it only requires each vehicle to use its own information $\boldsymbol{p}_i$, θ_i, $\boldsymbol{\eta}_i$ and $\dot{\boldsymbol{\eta}}_i$, and the information of its neighbors $\boldsymbol{\eta}_j$ and $\dot{\boldsymbol{\eta}}_j$, $j \in \bar{\mathcal{N}}_i$. For each vehicle i, sensors are required to measure $\boldsymbol{p}_i$ and θ_i, and communication devices are needed to obtain $\boldsymbol{\eta}_i$ and $\dot{\boldsymbol{\eta}}_i$, $j \in \bar{\mathcal{N}}_i$. For the vehicle which is the neighbor of the target, i.e., vehicle i, $i \in \bar{\mathcal{N}}_0$, it can measure $\boldsymbol{q}_0$ and $\boldsymbol{v}_0$ by sensors.

Now, the main result is presented as follows.

Theorem 1: Under Assumptions 1 and 2, the *cooperative moving-target enclosing control problem*, i.e., Problem 1, is solved by control law (9)–(10) with *first-order approximation* if there exists a constant $\delta \in (0, 1)$ and a $T \ge t_0$ such that

$$\|\boldsymbol{v}_0\|_{\infty,T} \le (1 - \delta)M \tag{12}$$

where $M = \min_{i=1}^{N} m_i$ is a constant with m_i defined in (11). That is, the trajectory of vehicles $\boldsymbol{p} := [\boldsymbol{p}_1^{\mathrm{T}}, ..., \boldsymbol{p}_N^{\mathrm{T}}]^{\mathrm{T}}$ converges to the *moving-target enclosing motion* $\check{\boldsymbol{p}}$ defined in Definition 1 with *first-order approximation*. Moreover, the constraint of bounded angular velocity (2) can always be satisfied. ■

Remark 3: The solvability condition (12) requires the target to move in a suitable velocity, which is of practical meaning. If the target is moving too fast, the vehicles are not able to track it while keeping a circular motion around it. Similar conditions were also assumed in [32, 33, 38–40]. To implement the proposed control law (9)–(10), each vehicle needs to measure its states in inertial frame by sensors and to obtain the information of its neighbors by communication devices, which is the same as that in [38]. However, compared with [38], the information of the target can only be known to some, not necessary all vehicles. ■

Remark 4: As the communication is assumed to be available in the network, a so-called "hopping" strategy described as follows can also be used. When vehicle k measures the information of the target $[\boldsymbol{q}_0^{\mathrm{T}}(t_s)\ \boldsymbol{v}_0^{\mathrm{T}}(t_s)]^{\mathrm{T}}$ at instant t_s, it transmits $[\boldsymbol{q}_0^{\mathrm{T}}(t_s)\ \boldsymbol{v}_0^{\mathrm{T}}(t_s)]^{\mathrm{T}}$ to its neighbor vehicle i, $i \in \bar{\mathcal{N}}_k$. After receiving the information from vehicle k, vehicle i transmits $[\boldsymbol{q}_0^{\mathrm{T}}(t_s)\ \boldsymbol{v}_0^{\mathrm{T}}(t_s)]^{\mathrm{T}}$ to vehicle j, $j \in \bar{\mathcal{N}}_i$. In the same manner, all vehicles will eventually obtain $[\boldsymbol{q}_0^{\mathrm{T}}(t_s)\ \boldsymbol{v}_0^{\mathrm{T}}(t_s)]^{\mathrm{T}}$. This "hopping" strategy was employed in [25], but has the following defect. Some vehicles have to wait long before they receive the information. These vehicles cannot obtain $[\boldsymbol{q}_0^{\mathrm{T}}(t_s)\ \boldsymbol{v}_0^{\mathrm{T}}(t_s)]^{\mathrm{T}}$ at an instant near t_s. Especially for a network consisting of large number of vehicles, the time delay caused by the "hopping" strategy will be large and cannot be neglected. However, our strategy does not require any vehicles to wait long. Each vehicles only needs to transmit its own information timely. As the time delay of the direct communication between two vehicles is usually small in practice, the time delay of the multi-vehicle systems can be ignored. ■

Remark 5: According to [53], let $\bar{\lambda}(\boldsymbol{k})$ denote the *log norm* of matrix $\boldsymbol{k}$ (associated with the 2-norm), i.e.,$\bar{\lambda}(\boldsymbol{k}) := \max\{\lambda|\lambda : \text{an eigenvalue of } (\boldsymbol{k} + \boldsymbol{k}^{\mathrm{T}})/2\}$. It follows from the design of $\boldsymbol{k}(\varepsilon_i)$, [53], and [54] that

$$\begin{aligned}\|e^{-\boldsymbol{k}(\varepsilon_i)t}\| &\le e^{\bar{\lambda}(-\boldsymbol{k}(\varepsilon_i))t} = e^{-\varepsilon_i t}\\ \|e^{\boldsymbol{k}(\varepsilon_i)t}\| &\le e^{\bar{\lambda}(\boldsymbol{k}(\varepsilon_i))t} = e^{\varepsilon_i t}.\end{aligned} \tag{13}$$

Moreover, the design of $\boldsymbol{k}(\varepsilon_i)$ make $\|\boldsymbol{k}(\varepsilon_i)\|$ satisfy

$$\|\boldsymbol{k}(\varepsilon_i)\| = \varepsilon_i,\ \ \|\boldsymbol{k}(\varepsilon_i)e^{-\boldsymbol{k}(\varepsilon_i)t}\| \le \varepsilon_i e^{-\varepsilon_i t}. \tag{14}$$

■

B. Stability Analysis

In what follows, the proof of Theorem 1, i.e., the stability analysis is presented in three steps. First, we show that the estimates $\boldsymbol{\eta}_i$ and $\dot{\boldsymbol{\eta}}_i$ can converge to $\boldsymbol{q}_0$ and $\boldsymbol{v}_0$ respectively. Then, we show that the trajectories of the closed-loop system consisting of system (7) and control law (9)–(10) can be approximated by an averaging system. Finally, we prove that the averaging system is globally exponentially stable.

1) Convergence of the Estimates: By the distributed update law (10), the estimates $\boldsymbol{\eta}_i$ and $\dot{\boldsymbol{\eta}}_i$ are able to converge to $\boldsymbol{q}_0$ and $\boldsymbol{v}_0$ respectively, as is stated as follows.

Proposition 1: Consider N systems (10), a target $\boldsymbol{q}_0$, and a given graph $\bar{\mathcal{G}}$. For any initial states $[\boldsymbol{\eta}_i^{\mathrm{T}}(t_0)\ \dot{\boldsymbol{\eta}}_i^{\mathrm{T}}(t_0)]^{\mathrm{T}} \in \mathbb{R}^4$, $i = 1, ..., N$, each $[\boldsymbol{\eta}_i^{\mathrm{T}}(t)\ \dot{\boldsymbol{\eta}}_i^{\mathrm{T}}(t)]^{\mathrm{T}}$ converges to $[\boldsymbol{q}_0^{\mathrm{T}}(t)\ \boldsymbol{v}_0^{\mathrm{T}}(t)]^{\mathrm{T}}$ exponentially as $t \to \infty$ under Assumptions 1 and 2. ∎

Proof: Define $\tilde{\boldsymbol{\eta}}_i = \boldsymbol{\eta}_i - \boldsymbol{q}_0$ and it follows from (10) that

$$\sum_{j\in\bar{\mathcal{N}}_i} a_{ij}(\dot{\tilde{\boldsymbol{\eta}}}_i - \dot{\tilde{\boldsymbol{\eta}}}_j) = \mu \sum_{j\in\bar{\mathcal{N}}_i} a_{ij}(\tilde{\boldsymbol{\eta}}_j - \tilde{\boldsymbol{\eta}}_i),\ j \in \bar{\mathcal{N}}_i. \tag{15}$$

Defining $\boldsymbol{\phi}_i = \sum_{j\in\bar{\mathcal{N}}_i} a_{ij}(\tilde{\boldsymbol{\eta}}_j - \tilde{\boldsymbol{\eta}}_i)$, (15) becomes $\dot{\boldsymbol{\phi}}_i = -\mu\boldsymbol{\phi}_i$. Then, $\boldsymbol{\phi}_i(t)$ converges to $\mathbf{0}$ exponentially as $t \to \infty$.

Denote $\boldsymbol{\phi} = \mathrm{col}(\boldsymbol{\phi}_1^{\mathrm{T}}, ..., \boldsymbol{\phi}_N^{\mathrm{T}})$ and $\tilde{\boldsymbol{\eta}} = \mathrm{col}(\boldsymbol{\eta}_1^{\mathrm{T}}, ..., \boldsymbol{\eta}_N^{\mathrm{T}})$, and it follows that

$$\boldsymbol{\phi} = (\mathcal{H} \otimes \boldsymbol{I}_2)\tilde{\boldsymbol{\eta}} \tag{16}$$

where $\mathcal{H}$ is a block matrix in the Laplacian matrix $\bar{\mathcal{L}}$ of the directed graph $\bar{\mathcal{G}}$. The Laplacian matrix $\bar{\mathcal{L}}$ can be partitioned as follows:

$$\bar{\mathcal{L}} = \left(\begin{array}{c|c} 0 & 0 \\ \hline -\mathcal{A}_0\mathbf{1}_N & \mathcal{H} \end{array}\right) \tag{17}$$

where $\mathcal{A}_0 = \mathrm{diag}(a_{10}, ..., a_{N0})$. Under Assumption 2, it follows from [49, Lemma 1] that $\mathcal{H}$ is a nonsingular matrix with all eigenvalues having positive real parts.

Since $\boldsymbol{\phi}(t)$ converges to $\mathbf{0}$ exponentially as $t \to \infty$, using (16) yields that $\tilde{\boldsymbol{\eta}}(t)$ converges to $\mathbf{0}$ exponentially as $t \to \infty$. Thus, each $\boldsymbol{\eta}_i(t)$ converges to $\boldsymbol{q}_0$ exponentially as $t \to \infty$.

Since $\dot{\boldsymbol{\phi}}(t)$ converges $\mathbf{0}$ exponentially as $t \to \infty$, it follows from (16) that $\dot{\boldsymbol{\phi}} = (\mathcal{H} \otimes \boldsymbol{I}_2)\dot{\tilde{\boldsymbol{\eta}}}$, and $\dot{\tilde{\boldsymbol{\eta}}}(t)$ converges to $\mathbf{0}$ exponentially as $t \to \infty$. Thus, each $\dot{\boldsymbol{\eta}}_i(t)$ converges to $\dot{\boldsymbol{q}}_0$, i.e., $\boldsymbol{v}_0(t)$ exponentially as $t \to \infty$.

The proof is thus completed. ∎

2) Approximation of an Averaging System: The closed-loop system consisting of system (7) and control law (9)–(10) can be written as

$$\begin{aligned} \dot{\boldsymbol{q}}_i &= -\boldsymbol{b}(\theta_i)\boldsymbol{f}_i(\boldsymbol{q}_i, \boldsymbol{\eta}_i) \\ \dot{\theta}_i &= \omega_i \end{aligned} \tag{18}$$

where

$$\begin{aligned} \boldsymbol{b}(\theta_i) &= \begin{bmatrix} \cos^2\theta_i & \cos\theta_i\sin\theta_i \\ \sin\theta_i\cos\theta_i & \sin^2\theta_i \end{bmatrix} \\ \boldsymbol{f}_i(\boldsymbol{q}_i, \boldsymbol{\eta}_i) &= \boldsymbol{\sigma}_i(\boldsymbol{k}(\varepsilon_i)(\boldsymbol{q}_i - \boldsymbol{\eta}_i) - \dot{\boldsymbol{\eta}}_i). \end{aligned} \tag{19}$$

Motivated by [29], we adopt an approximation method and then obtain an average system for system (18), so as to tackle the nonlinearity of system (18). The average system for system (18) can be obtained by the following proposition.

Proposition 2: Consider the closed-loop system (18) with the parameter r_i fixed and v_i sufficiently large. Define $\bar{T} = \frac{2\pi r_i}{v_i}$ and an average state

$$\bar{\boldsymbol{q}}_i(t) = \frac{1}{\bar{T}}\int_t^{t+\bar{T}} \boldsymbol{q}_i(\tau)\mathrm{d}\tau,\ t \geq t_0. \tag{20}$$

Then, for any $t \geq t_0$, we have

$$\bar{\boldsymbol{q}}_i(t) = \boldsymbol{q}_i(t) + \boldsymbol{O}(\frac{r_i}{v_i}) \tag{21}$$

$$\dot{\bar{\boldsymbol{q}}}_i = -\boldsymbol{f}_i(\bar{\boldsymbol{q}}_i, \boldsymbol{\eta}_i) + \boldsymbol{O}(\frac{r_i}{v_i}) \tag{22}$$

where $\boldsymbol{O}(\frac{r_i}{v_i})$ represents the first order of smallness as $\frac{r_i}{v_i} \to 0$. ∎

Proof: Since $\boldsymbol{f}_i(\cdot)$ is a bounded function, $\|\boldsymbol{f}_i(\bar{\boldsymbol{q}}_i, \boldsymbol{\eta}_i)\|$ and $\dot{\boldsymbol{q}}_i$ are bounded. Denote $\|\dot{\boldsymbol{q}}_i\| \leq \mu_1 \leq \infty$, and then we have

$$\begin{aligned} \|\boldsymbol{q}_i(\tau) - \boldsymbol{q}_i(t)\| &= \|\int_t^\tau \dot{\boldsymbol{q}}_i(s)\mathrm{d}s\| \leq \mu_1(\tau - t) \\ &\leq \mu_1\bar{T} = \frac{2\mu_1\pi r_i}{v_i},\ \forall\tau \in [t, t+\bar{T}]. \end{aligned} \tag{23}$$

For the smooth trajectory $\boldsymbol{q}_i(s)$, $s \in [t, t+\bar{T}]$, there exists some instant τ such that $\boldsymbol{q}_i(\tau) = \frac{1}{\bar{T}}\int_t^{t+\bar{T}}\boldsymbol{q}_i(s)\mathrm{d}s$, which together with (23) implies that $\|\bar{\boldsymbol{q}}_i(t) - \boldsymbol{q}_i(t)\| \leq \frac{2\mu_1\pi r_i}{v_i}$ or $\bar{\boldsymbol{q}}_i(t) - \boldsymbol{q}_i(t) = \boldsymbol{O}(\frac{r_i}{v_i})$. Thus, (21) is obtained.

Next, it follows from (19) that

$$\begin{aligned} &\|\boldsymbol{f}_i(\boldsymbol{q}_i(\tau), \boldsymbol{\eta}_i(\tau)) - \boldsymbol{f}_i(\boldsymbol{q}_i(t), \boldsymbol{\eta}_i(t))\| \\ &\quad\leq \mu_2\|\boldsymbol{k}(\varepsilon_i)(\boldsymbol{q}_i(\tau) - \boldsymbol{q}_i(t)) - \boldsymbol{k}(\varepsilon_i)(\boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_i(t)) \\ &\quad\quad - (\dot{\boldsymbol{\eta}}_i(\tau) - \dot{\boldsymbol{\eta}}_i(t))\| \\ &\quad\leq \mu_3\|\boldsymbol{q}_i(\tau) - \boldsymbol{q}_i(t)\| + \mu_4\|\boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_i(t)\| \\ &\quad\quad + \mu_5\|\dot{\boldsymbol{\eta}}_i(\tau) - \dot{\boldsymbol{\eta}}_i(t)\| \end{aligned} \tag{24}$$

for some positive constants μ_2, μ_3, μ_4, and μ_5. By Proposition 1 and Assumption 1, $\dot{\boldsymbol{\eta}}_i$ and $\dot{\boldsymbol{\eta}}_i$ are bounded. Then, $\mu_4\|\boldsymbol{\eta}_i(\tau) - \boldsymbol{\eta}_i(t)\| + \mu_5\|\dot{\boldsymbol{\eta}}_i(\tau) - \dot{\boldsymbol{\eta}}_i(t)\| \leq \frac{2\mu_6\pi r_i}{v_i}$ for some positive constant μ_6, which together with $\boldsymbol{q}_i(\tau) - \boldsymbol{q}_i(t) = \boldsymbol{O}(\frac{r_i}{v_i})$ implies that

$$\boldsymbol{f}_i(\boldsymbol{q}_i(\tau), \boldsymbol{\eta}_i(\tau)) - \boldsymbol{f}_i(\boldsymbol{q}_i(t), \boldsymbol{\eta}_i(t)) = \boldsymbol{O}(\frac{r_i}{v_i}). \tag{25}$$

Thus, with $\bar{\boldsymbol{q}}_i(t) - \boldsymbol{q}_i(t) = \boldsymbol{O}(\frac{r_i}{v_i})$, we have

$$\boldsymbol{f}_i(\bar{\boldsymbol{q}}_i(t), \boldsymbol{\eta}_i(t)) - \boldsymbol{f}_i(\boldsymbol{q}_i(t), \boldsymbol{\eta}_i(t)) = \boldsymbol{O}(\frac{r_i}{v_i}). \tag{26}$$

Furthermore, we have the following calculation:

$$\begin{aligned} \dot{\bar{\boldsymbol{q}}}_i(t) &= \frac{1}{\bar{T}}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^{t+\bar{T}} \boldsymbol{q}_i(\tau)\mathrm{d}\tau = \frac{1}{\bar{T}}\int_t^{t+\bar{T}} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}_i(\tau)\mathrm{d}\tau \\ &= -\frac{1}{\bar{T}}\int_t^{t+\bar{T}} \boldsymbol{b}(\theta(\tau))\boldsymbol{f}_i(\boldsymbol{q}_i(\tau), \boldsymbol{\eta}_i(\tau))\mathrm{d}\tau \\ &= -\frac{1}{\bar{T}}\int_t^{t+\bar{T}} \boldsymbol{b}(\theta(\tau))\mathrm{d}\tau(\boldsymbol{f}_i(\bar{\boldsymbol{q}}_i(t) + \boldsymbol{O}(\frac{r_i}{v_i}), \boldsymbol{\eta}_i(t))) + \boldsymbol{O}(\frac{r_i}{v_i})) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\bar{T}}\int_{t}^{t+\bar{T}} \boldsymbol{b}(\theta(\tau))\mathrm{d}\tau(\boldsymbol{f}_i(\bar{\boldsymbol{q}}_i(t),\boldsymbol{\eta}_i(t)) + \boldsymbol{O}(\frac{r_i}{v_i}))\\
&= -\boldsymbol{f}_i(\bar{\boldsymbol{q}}_i(t),\boldsymbol{\eta}_i(t)) + \boldsymbol{O}(\frac{r_i}{v_i}) \qquad (27)
\end{aligned}
$$

where it is noted that $\theta(\tau) = \omega_i\tau$ and $\frac{1}{\bar{T}}\int_t^{t+\bar{T}} \boldsymbol{b}(\frac{v_i}{r_i}\tau)\mathrm{d}\tau = \boldsymbol{I}_2$.

The proof is thus completed. ■

Denote $\boldsymbol{q} = \mathrm{col}(\boldsymbol{q}_1^{\mathrm{T}},...,\boldsymbol{q}_N^{\mathrm{T}})$, $\boldsymbol{\theta} = \mathrm{col}(\theta_1,...,\theta_N)$, $\boldsymbol{\eta} = \mathrm{col}(\boldsymbol{\eta}_1^{\mathrm{T}},...,\boldsymbol{\eta}_N^{\mathrm{T}})$, $\boldsymbol{B}(\boldsymbol{\theta}) = \mathrm{diag}(\boldsymbol{b}(\theta_i),...,\boldsymbol{b}(\theta_N))$, $\boldsymbol{F}(\boldsymbol{q},\boldsymbol{\eta}) = \mathrm{col}(\boldsymbol{f}_1(\boldsymbol{q}_1,\boldsymbol{\eta}_1),...,\boldsymbol{f}_N(\boldsymbol{q}_N,\boldsymbol{\eta}_N))$. Then, the closed-loop system consisting of N systems (18) can be written in form of

$$\dot{\boldsymbol{q}} = -\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{F}(\boldsymbol{q},\boldsymbol{\eta}). \qquad (28)$$

If the smallness $\boldsymbol{O}(\frac{r_i}{v_i})$ in (21) and (22) is ignored, we have

$$\dot{\boldsymbol{q}}_i = -\boldsymbol{f}_i(\boldsymbol{q}_i,\boldsymbol{\eta}_i). \qquad (29)$$

Then, the system consisting of N systems (29) is written as

$$\dot{\boldsymbol{q}} = -\boldsymbol{F}(\boldsymbol{q},\boldsymbol{\eta}) \qquad (30)$$

which is called the *average system* of system (28). That is, by ignoring the smallness $\boldsymbol{O}(\frac{r_i}{v_i})$ in (21) and (22), the trajectory of system (28) can be approximated by that of system (30).

As in [29], we establish the stability analysis on the approximate system (28), and then run simulation on the original system (30) to illustrate the effectiveness of this approach.

3) Stability of the Averaging System: To prove Theorem 1, it remains to show that the trajectory of the average system (30) can converge to $\boldsymbol{1}_N \otimes \boldsymbol{q}_0(t)$ as $t\to\infty$ for any initial state $\boldsymbol{q}(t_0)$, $\forall t_0 \geq 0$.

To this end, consider the tracking error defined as

$$\tilde{\boldsymbol{q}}_i = \boldsymbol{q}_i - \boldsymbol{q}_0. \qquad (31)$$

Based on (29), the error dynamics are obtained as follows:

$$\dot{\tilde{\boldsymbol{q}}}_i = -\boldsymbol{\sigma}_i(\boldsymbol{k}(\varepsilon_i)(\tilde{\boldsymbol{q}}_i - \tilde{\boldsymbol{\eta}}_i) - \dot{\tilde{\boldsymbol{\eta}}}_i - \boldsymbol{v}_0) - \boldsymbol{v}_0. \qquad (32)$$

If the saturation element in $\boldsymbol{\sigma}_i(\cdot)$ is nonexistent, system (32) becomes

$$\dot{\tilde{\boldsymbol{q}}}_i = -\boldsymbol{k}(\varepsilon_i)(\tilde{\boldsymbol{q}}_i - \tilde{\boldsymbol{\eta}}_i) - \dot{\tilde{\boldsymbol{\eta}}}_i. \qquad (33)$$

In this case, the following proposition holds.

Proposition 3: System (33) is globally asymptotically stable at $\tilde{\boldsymbol{q}}_i = 0$ if Proposition 1 holds. ■

Proof: See Appendix A. ■

Next, we show that the saturation element in $\boldsymbol{\sigma}_i(\cdot)$ will be nonexistent after some instant, and system (32) will operate linearly as system (33) from then on.

By condition (12), there exists an instant T_1 such that

$$\|\boldsymbol{v}_0\|_{\infty,T_1} \leq (1-\delta)M,\ 0<\delta<1. \qquad (34)$$

By Proposition 1, $[\tilde{\boldsymbol{\eta}}_i(t)\ \dot{\tilde{\boldsymbol{\eta}}}_i(t)]^{\mathrm{T}}$ converges to $[\boldsymbol{q}_0(t)\ \boldsymbol{v}_0(t)]^{\mathrm{T}}$ exponentially. Then, under Assumption 1, there exists an instant $T_2 \geq T_1$ such that for all $\tilde{\boldsymbol{\eta}}_i(t_0)$ and $\dot{\tilde{\boldsymbol{\eta}}}_i(t_0)$,

$$\|\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i\|_{\infty,T_2} \leq \frac{\delta M}{3},\ \|\dot{\tilde{\boldsymbol{\eta}}}_i\|_{\infty,T_2} \leq \frac{\delta M}{3}. \qquad (35)$$

It can be observed from (32) that the trajectory of $\tilde{\boldsymbol{q}}_i(t)$ is determined by a linear differential equation with two bounded functions $\boldsymbol{\sigma}_i$ and $\boldsymbol{v}_0$, i.e.,

$$
\begin{aligned}
\tilde{\boldsymbol{q}}_i(T_2) =& \tilde{\boldsymbol{q}}_i(t_0) + \int_{t_0}^{T_2}(\boldsymbol{\sigma}_i(\boldsymbol{k}(\varepsilon_i)(\tilde{\boldsymbol{q}}_i(\tau) - \tilde{\boldsymbol{\eta}}_i(\tau))\\
&- \dot{\tilde{\boldsymbol{\eta}}}_i(\tau) - \boldsymbol{v}_0(\tau)) - \boldsymbol{v}_0(\tau))\mathrm{d}\tau. \qquad (36)
\end{aligned}
$$

Thus, for an arbitrary $\tilde{\boldsymbol{q}}_i(t_0)$, $\tilde{\boldsymbol{q}}_i(T_2)$ is bounded independent of ε_i, and there exists a constant $\beta_1 > 0$ associated with $\tilde{\boldsymbol{q}}_i(t_0)$ such that

$$\|\tilde{\boldsymbol{q}}_i(T_2)\| \leq \beta_1. \qquad (37)$$

Assume that the saturation element in $\boldsymbol{\sigma}_i(\cdot)$ is nonexistent from T_2 onwards. Then, system (32) becomes (33). Note that $\tilde{\boldsymbol{\eta}}_i(t)$ converges to $\boldsymbol{0}$ exponentially with a rate independent of ε_i. It follows from (14) and (33) that for a given ε_i, there exist a constant $\beta_2 > 0$ such that

$$\int_{T_2}^{\infty} \|e^{\varepsilon_i\tau}(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i(\tau) - \dot{\tilde{\boldsymbol{\eta}}}_i(\tau))\|\mathrm{d}\tau \leq \beta_2. \qquad (38)$$

Then, using (13) and (14), we have the following calculation for any instant $t \geq T_2$,

$$
\begin{aligned}
\|\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{q}}_i(t)\| =& \|\boldsymbol{k}(\varepsilon_i)(e^{-\boldsymbol{k}(\varepsilon_i)(t-T_2)}\tilde{\boldsymbol{q}}_i(T_2))\\
&+ \int_{T_2}^{t} e^{-\boldsymbol{k}(\varepsilon_i)(t-\tau)}(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}(\tau) - \dot{\tilde{\boldsymbol{\eta}}}_i(\tau))\mathrm{d}\tau\|\\
\leq& \beta_1\varepsilon_i e^{-\varepsilon_i(t-T_2)} + \varepsilon_i e^{-\varepsilon_i(t-T_2)}\\
&\times \int_{T_2}^{t} \|e^{\boldsymbol{k}(\varepsilon_i)\tau}(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i(\tau) - \dot{\tilde{\boldsymbol{\eta}}}_i(\tau))\|\mathrm{d}\tau\\
\leq& \beta_1\varepsilon_i e^{-\varepsilon_i(t-T_2)} + \varepsilon_i e^{-\varepsilon_i(t-T_2)}\\
&\times \int_{T_2}^{t} \|e^{\varepsilon_i\tau}(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i(\tau) - \dot{\tilde{\boldsymbol{\eta}}}_i(\tau))\|\mathrm{d}\tau\\
\leq& \beta_1\varepsilon_i e^{-\varepsilon_i(t-T_2)} + \beta_2\varepsilon_i e^{-\varepsilon_i(t-T_2)}\\
\leq& (\beta_1+\beta_2)\varepsilon_i. \qquad (39)
\end{aligned}
$$

If $(\beta_1+\beta_2)\varepsilon_i < \frac{\delta M}{3}$, then

$$\|\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{q}}_i\|_{\infty,T_2} < \frac{\delta M}{3}. \qquad (40)$$

By combining (34), (35), and (40), we have

$$
\begin{aligned}
&\|\boldsymbol{k}(\varepsilon_i)(\tilde{\boldsymbol{q}}_i - \tilde{\boldsymbol{\eta}}_i) - \dot{\tilde{\boldsymbol{\eta}}}_i - \boldsymbol{v}_0\|_{\infty,T_2}\\
&< \frac{\delta M}{3} + \frac{\delta M}{3} + \frac{\delta M}{3} + (1-\delta)M = M, \qquad (41)
\end{aligned}
$$

which shows that system (32) will operate linearly after instant T_2.

If $(\beta_1+\beta_2)\varepsilon_i \geq \frac{\delta M}{3}$, it is possible that system (32) will not operate linearly in some instants after T_2 if the following two cases happen. To analyze these two cases, we denote $\tilde{\boldsymbol{q}}_i = [\tilde{x}_i\ \tilde{y}_i]^{\mathrm{T}}$ and $\boldsymbol{v}_0 = [v_0^x\ v_0^y]^{\mathrm{T}}$, and consider the dynamics of $\dot{\tilde{x}}_i$ and $\dot{\tilde{y}}_i$ separately.

In the first case, there is an instant $T_3 > T_2$ such that $[1\ 0]\boldsymbol{\sigma}_i(T_3) = m_i$ and then $\dot{\tilde{x}}_i(T_3) = -m_i - v_0^x(T_3)$, which implies that $\tilde{x}_i(T_3) \geq \frac{\delta m_i}{3} \geq \frac{\delta M}{3} > 0$. Since $|v_0^x| < M \leq m_i$, then $\dot{\tilde{x}}_i(T_3) < 0$ and $\tilde{x}_i(t)$, $t > T_3$, will decrease.

In the second case, there is an instant $T_3 > T_2$ such that $[1\ 0]\boldsymbol{\sigma}_i(T_3) = -m_i$ and then $\dot{\tilde{x}}_i(T_3) = m_i - v_0^x(T_3)$, which implies that $\tilde{x}_i(T_3) \leq -\frac{\delta m_i}{3} \leq -\frac{\delta M}{3} < 0$. Since $|v_0^x| < M \leq$

m_i, then $\dot{\tilde{x}}_i(T_3) > 0$ and $\tilde{x}_i(t)$, $t > T_3$, will increase.

In both case, $|\tilde{x}_i(t)|$ will decrease. As $|\tilde{x}_i(t)|$, $t > T_3$, maintain decreasing, there must exist an instant $T_4 \geq T_3$ such that $\|\varepsilon_i \tilde{x}_i\|_{\infty,T_4} < \frac{\sqrt{2}\delta M}{6}$.

Using the same analysis above, we can also obtain that there exist an instant $T_5 \geq T_4$ such that $\|\varepsilon_i \tilde{y}_i\|_{\infty,T_5} < \frac{\sqrt{2}\delta M}{6}$.

Thus, we have $\|\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{q}}_i\|_{\infty,T_5} < \frac{\delta M}{3}$, and then

$$\|\boldsymbol{k}(\varepsilon_i)(\tilde{\boldsymbol{q}}_i - \tilde{\boldsymbol{\eta}}_i) - \dot{\tilde{\boldsymbol{\eta}}}_i - \boldsymbol{v}_0\|_{\infty,T_5} < M, \tag{42}$$

which indicates that system (32) will operate linearly after instant T_5. By Proposition 3, $\tilde{\boldsymbol{q}}_i(t)$ converges to $\mathbf{0}$ exponentially as $t \to \infty$.

Finally, it follows from (9) that the angular velocity ω_i satisfies the following inequality

$$\frac{v_i}{r_i} - m_i(\sin\theta_i + \cos\theta_i) \leq \omega_i \leq \frac{v_i}{r_i} + m_i(\sin\theta_i + \cos\theta_i).$$

Then, with m_i selected by (11), the constraint of bounded angular velocity (2) can always be satisfied.

Hence, the proof of Theorem 1 is completed. ∎

C. Discussions

The main difficulty of making the tracking error $\tilde{\boldsymbol{q}}_i(t) = \boldsymbol{q}_i(t) - \boldsymbol{q}_0(t)$, $i = 1, .., N$, converge to $\mathbf{0}$ lies in the fact that the trajectory of $\boldsymbol{q}_i(t)$ is in a 2-D plane but is controlled by one input channel, i.e., $v_i - \omega_i r_i$. This basic feature can be observed from (7), and is essentially due to the nonholonomic constraint of the unicycle model, i.e., $[\sin\theta_i \ \ -\cos\theta_i]\dot{\boldsymbol{p}}_i = 0$. To tackle the term $[\cos\theta_i \ \ \sin\theta_i]^{\mathrm{T}}$ in system (7), we have added the term $[\cos\theta_i \ \ \sin\theta_i]$ in controller (9) and $\boldsymbol{b}(\theta_i)$ in system (18) is thus obtained. Based on (18), the error dynamics are

$$\dot{\tilde{\boldsymbol{q}}}_i = -\boldsymbol{b}(\theta_i)\boldsymbol{\sigma}_i(\boldsymbol{k}(\varepsilon_i)(\tilde{\boldsymbol{q}}_i - \tilde{\boldsymbol{\eta}}_i) - \dot{\tilde{\boldsymbol{\eta}}}_i - \boldsymbol{v}_0) - \boldsymbol{v}_0. \tag{43}$$

Then, in order to deal with the nonlinearity of $\boldsymbol{b}(\theta_i)$, we adopt an averaging system method used in [29].

We have proved that the error dynamics of the average system (30) is exponentially stable at $[\tilde{\boldsymbol{q}}_i^{\mathrm{T}} \ \ \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{0}$. While $[\tilde{\boldsymbol{q}}_i^{\mathrm{T}} \ \ \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{0}$ is not the equilibrium for the original error dynamics (43), as system (43) becomes $\dot{\tilde{\boldsymbol{q}}}_i = \boldsymbol{b}(\theta_i)\boldsymbol{v}_0 - \boldsymbol{v}_0$ when $[\tilde{\boldsymbol{q}}_i^{\mathrm{T}} \ \ \tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{0}$. As shown in Proposition 2, the trajectory of the average system (30) can approximate that of the original system (28) with the first-order smallness $\boldsymbol{O}(\frac{r_i}{v_i})$. To minimize the first-order smallness $\boldsymbol{O}(\frac{r_i}{v_i})$, we can select a large v_i or a small r_i, and meanwhile ensure $\omega_i^{\max} r_i > v_i$. In practice, if vehicles are allowed to stay close to the given target, radii r_i can be set as small as possible. Once the radii r_i are set, the constant linear velocities v_i need to be as large as possible. Besides, Theorem 1 indicates that the velocity of the given target needs to be properly bounded as (12). In fact, if vehicles have much larger linear velocities than the velocity of the target, they will have more allowance of velocities to orbit around the target in addition to tracking the target. Hence, in practice, it is desirable for the vehicle to be capable of a large linear velocity.

Based on the stability analysis presented above, we have the following corollaries.

Corollary 1: Consider error dynamics (43), the trajectory of the tracking error $\tilde{\boldsymbol{q}}_i$ is globally uniformly ultimately bounded if Assumptions 1–2 and condition (12) are satisfied. ∎

Corollary 2: The error dynamics (43) is globally uniformly asymptotically stable at $\tilde{\boldsymbol{q}}_i = \mathbf{0}$ if Assumptions 1–2 and condition (12) are satisfied, and there exists an instant T' such that $\|\boldsymbol{v}_0\|_{\infty,T'} = 0$. ∎

Sketch of Proof: Based on the proof of Theorem 1, if Assumptions 1–2 and condition (12) are satisfied, there exists an instant T_2 such that system (43) will operate without the saturation element after T_2 and becomes

$$\dot{\tilde{\boldsymbol{q}}}_i = -\boldsymbol{b}(\theta_i)\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{q}}_i + \boldsymbol{b}(\theta_i)(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i + \dot{\tilde{\boldsymbol{\eta}}}_i + \boldsymbol{v}_0) - \boldsymbol{v}_0. \tag{44}$$

First, consider the case where there exists an instant T' such that $\|\boldsymbol{v}_0\|_{\infty,T'} = 0$. In this case, after an instant $T_6 = \max\{T', T_5\}$, system (44) can be written as a nominal system

$$\dot{\tilde{\boldsymbol{q}}}_i = -\boldsymbol{b}(\theta_i)\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{q}}_i \tag{45}$$

with a perturbation $\boldsymbol{b}(\theta_i)(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i + \dot{\tilde{\boldsymbol{\eta}}}_i)$. It follows from Proposition 1 that this perturbation converges to $\mathbf{0}$ exponentially. For the nominal system (45), consider a Lyapunov function candidate $V(\tilde{\boldsymbol{q}}_i) = \frac{1}{2}\tilde{\boldsymbol{q}}_i^{\mathrm{T}}\tilde{\boldsymbol{q}}_i$. The time derivative of $V(\tilde{\boldsymbol{q}}_i)$ along the trajectory of system (45) can be obtained as

$$\dot{V}(\tilde{\boldsymbol{q}}_i) = -\tilde{\boldsymbol{q}}_i^{\mathrm{T}}\boldsymbol{b}(\theta_i)\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{q}}_i \tag{46}$$

where it is noted that $\boldsymbol{k}(\varepsilon_i)$ is a scalar matrix with all diagonal elements positive. Since $\boldsymbol{b}(\theta_i)$ is a symmetric matrix of which the eigenvalues are always 1 and 0 irrespective of θ_i, we have $\dot{V}(\tilde{\boldsymbol{q}}_i) \leq 0$. Then, system (45) is globally uniformly stable. As $V(\tilde{\boldsymbol{q}}_i)$ is nonincreasing in t and bounded, $\lim_{t\to\infty}\int_0^t \dot{V}(\tilde{\boldsymbol{q}}_i(\tau))\mathrm{d}\tau$ exists and is finite, and $\tilde{\boldsymbol{q}}_i$ is bounded. Since ω_i is bounded, $\ddot{V}(\tilde{\boldsymbol{q}}_i)$ is bounded and $\dot{V}(\tilde{\boldsymbol{q}}_i)$ is uniformly continuous in t. By Barbalat's Lemma, $\lim_{t\to\infty}\tilde{\boldsymbol{q}}_i(t) = \mathbf{0}$. Thus, the nominal system (45) is globally uniformly asymptotically stable. Finally, by using [55, Lemma 2.1], Corollary 2 can be obtained.

Furthermore, system (44) is essentially a nominal system

$$\dot{\tilde{\boldsymbol{q}}}_i = -\boldsymbol{b}(\theta_i)\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{q}}_i + \boldsymbol{b}(\theta_i)(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i + \dot{\tilde{\boldsymbol{\eta}}}_i) \tag{47}$$

with a perturbation $\boldsymbol{b}(\theta_i)\boldsymbol{v}_0 - \boldsymbol{v}_0$. Using the proof of Corollary 2, we can prove that the nominal system (47) is globally uniformly asymptotically stable. The perturbation $\boldsymbol{b}(\theta_i)\boldsymbol{v}_0 - \boldsymbol{v}_0$ is bounded under Assumption 1, and is nonvanishing if $\boldsymbol{v}_0(t)$ does not converge to $\mathbf{0}$. Then, similar to the proof of [55, Lemma 2.1], it can be obtained that $\tilde{\boldsymbol{q}}_i$ is globally uniformly ultimately bounded, as stated in Corollary 1. ∎

It is obvious the ultimate bound will be smaller as the bound of $\boldsymbol{v}_0$ is smaller. For a given target velocity $\boldsymbol{v}_0$, Theorem 1 indicates that minimizing the first order smallness $\boldsymbol{O}(\frac{r_i}{v_i})$ can still reduce the ultimate bound. This fact can be observed by the following analysis. It follows from system (44) that $\tilde{\boldsymbol{q}}_i(t) = \psi_i(t, T_2')\tilde{\boldsymbol{q}}_i(T_2') + \int_{T_2'}^t \psi_i(t,\tau)(\boldsymbol{b}(\theta_i(\tau))(\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i(\tau) + \dot{\tilde{\boldsymbol{\eta}}}_i(\tau) + \boldsymbol{v}_0(\tau)) - \boldsymbol{v}_0(\tau))\mathrm{d}\tau$, $\forall t \geq T_2' \geq T_2$, where $\psi_i(t, T_2')$ is the state transition matrix corresponding to system matrix $\boldsymbol{b}(\theta_i(t))\boldsymbol{k}(\varepsilon_i)$, and instant T_2' is selected such that $\|\tilde{\boldsymbol{\eta}}_i(t)\|$ and $\|\dot{\tilde{\boldsymbol{\eta}}}_i(t)\|$, $\forall t \geq T_2'$, are sufficiently small since $\tilde{\boldsymbol{\eta}}_i(t)$ and $\dot{\tilde{\boldsymbol{\eta}}}_i(t)$ exponentially converge to zero as $t \to \infty$ by Proposition 1. By system (45) and Corollary 2, we can obtain

$\psi_i(t, T_2)\tilde{\boldsymbol{q}}_i(T_2)$, $\forall t \geq T_2$, converges to zero as $t \to \infty$. Thus, the bound of the track error $\tilde{\boldsymbol{q}}_i$ is determined by $\int_{T_2'}^{t} \psi_i(t,\tau)(\boldsymbol{b}(\theta_i(\tau)) - \boldsymbol{I}_2)\boldsymbol{v}_0(\tau)\mathrm{d}\tau$. Note that $\theta(\tau) = \omega_i\tau$ and $\frac{1}{\bar{T}}\int_{t}^{t+\bar{T}} \boldsymbol{b}(\frac{v_i}{r_i}\tau)\mathrm{d}\tau = \boldsymbol{I}_2$, where $\bar{T}$ is defined as $\bar{T} = \frac{2\pi r_i}{v_i}$. If $\frac{r_i}{v_i}$ is sufficient small, then $\int_{T_2'}^{t} \psi_i(t,\tau)(\boldsymbol{b}(\theta_i(\tau)) - \boldsymbol{I}_2)\boldsymbol{v}_0(\tau)\mathrm{d}\tau$ is small. If $\frac{r_i}{v_i}$ is large, we may use $2\|\boldsymbol{v}_0\|_{\infty,T}$ (the upper bound of $\|(\boldsymbol{b}(\theta_i(t)) - \boldsymbol{I}_2)\boldsymbol{v}_0(t)\|$) to compute the ultimate bound of the track error $\tilde{\boldsymbol{q}}_i$. Moreover, Corollary 2 implies that for the stationary-target case, the proposed control law can make the tracking error $\tilde{\boldsymbol{q}}_i(t)$ converge to $\mathbf{0}$.

Remark 6: Although there exist approaches on distributed control of multi-agent systems or target enclosing control of unicycle-type vehicles, yet these approaches cannot be directly applied or easily extended to the particular problem studied in this paper. This fact can be observed by the following three aspects. *First*, the cooperative moving-target enclosing control problem of multi-agent systems is more complicated than typical distributed control problems of multi-agent systems, such as the leaderless/leader-following consensus and formation [1–4]. Several existing works have studied this problem for multi-agent systems with simple linear dynamics, for instance multiple single- or double-integrators [43–46]. *Second*, the cooperative target enclosing control problem becomes more difficult for multiple unicycle-type vehicles. Most existing works on unicycle-type vehicles only studied the stationary-target enclosing control problem, as reviewed in Section I. But the case with a stationary target is only a special case of the one with a moving target. Besides, most existing approaches for the case with a stationary target assumed the position of the target is known to all vehicles, for instance [20, 23, 25–28]. While Corollary 2 shows that our proposed control law makes vehicles enclose the stationary target under the assumption that only some of the vehicles know the position of the target. *Third*, the cooperative target enclosing control problem becomes more challenging if the vehicles are unicycle-type and the target moves with a time-varying velocity, as pointed out in [37–40]. To solve this particular problem, all existing approaches rely on the *global* knowledge of the information of the target. In particular, for a target with a constant velocity, all vehicles have to know its position [34–36], or its position and velocity [32, 33]. For a target with a time-varying velocity, all vehicles have to know its position, velocity, and acceleration in [38–40]. We consider the assumption that only some of the vehicles (at least one) know the information of the target, and make the first attempt to solve this problem under this assumption. Once some vehicles need to estimate the information of the target, the stability of the resulting closed-loop multi-vehicle system becomes more complicated. It turns out not easy to find suitable Lyapunov function candidates as the existing works did. Motivated by [29], we use the averaging system theory and analyze the trajectory of the averaging system, which is not a typical method in the distributed control problems. ■

Remark 7: The proposed control law is distributed at the price of resulting in globally uniformly ultimately bounded tracking error. While in [37], only the local one was achieved. Similar to [41], once the target is moving with a specified velocity $\boldsymbol{v}_0$, the ultimate bound is associated with v_i and r_i. In the case where v_i and r_i are fixed, the ultimate bound is associated with $\boldsymbol{v}_0$. It follows from the proposed control law (9)–(10) that the steady-state angular velocity is time-varying and not a fixed value. Our approach is to use the trajectory of an averaging system to approximate that of the original closed-loop system with a first-order smallness, and then to prove the averaging system is asymptotically stable. Thus, as the tracking error in the steady state is not zero, the steady-state angular velocity is not constant. Since v_i and r_i are constant, and the velocity of the target $\boldsymbol{v}_0$ is time-varying, the steady-state angular velocity must be time-varying so as to orbit around the target while tracking the moving target. ■

Remark 8: This paper focuses on the case where the linear velocities of vehicles maintain constant and are not the control channels. We have used the angular velocity as the only control channel for each vehicle, to deal with two objectives. The first one is to make vehicles orbit around the target, and the second one is to let the circular motion center of each vehicle track the moving target. In fact, it is challenging to further achieve a regular pattern by only using the angular velocity as the control channel. To form a regular pattern, it may require to use the linear velocity as the control channel, which is one of the future topics. Moreover, if the vehicles are in the different altitudes, the vehicles will not collide or coincide. Otherwise, they may collide or coincide if the vehicles has the same linear velocities and desired radii. In order to avoid this, we can incorporate our proposed control law with the behavior-based algorithm implemented in the experiment of [28], and use a switched law similar to that in [28]. Based on the behavior-based algorithm in [28], when two vehicles are going to collide and satisfy a certain condition in the algorithm, they switch to the controller for avoiding collision used in [28]. Then, they switch back to our proposed control law once they do not tend to collide. It should be noted that our proposed control law allows each vehicle to select its own initial time, and the obtained result is independent of the selection on initial time, which facilitates the implementation with the switch law based on the behavior-based algorithm in [28]. ■

IV. Simulation Example

In this section, the main result is illustrated by an simulation example, where five vehicles (1) are required to enclose a moving target. The values of variables in this example are in SI units, and the units are omitted for convenience.

The topology of the network among the target (labeled 0) and five vehicles (labeled 1–5) is described by a directed graph $\bar{\mathcal{G}}$ shown in Fig. 2, which satisfies Assumption 2. The constraints of bounded angular velocity for each vehicle are given by (2) with $\omega_1^{\max} = 1.5$, $\omega_2^{\max} = 1.55$, $\omega_3^{\max} = 1.6$, $\omega_4^{\max} = 1.4$, and $\omega_5^{\max} = 1.45$. The linear velocities of each vehicle are given by $v_1 = 2$, $v_2 = 2.5$, $v_3 = 3$, $v_4 = 3.5$, and $v_5 = 4$. The radius of each vehicle are given by $r_1 = 4$, $r_2 = 5$, $r_3 = 6$, $r_4 = 7$, and $r_5 = 8$, which satisfies $r_i > \frac{v_i}{\omega_i^{\max}}$.

The initial states of all vehicles are given by $\boldsymbol{p}_1(t_0) = [-10\ 5]^{\mathrm{T}}$, $\theta_1(t_0) = \pi$, $\boldsymbol{p}_2(t_0) = [-5\ \ -10]^{\mathrm{T}}$, $\theta_2(t_0) = -\frac{5}{6}\pi$,

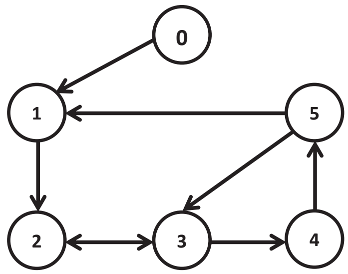

Fig. 2. The topology of the sensor graph $\bar{\mathcal{G}}$.

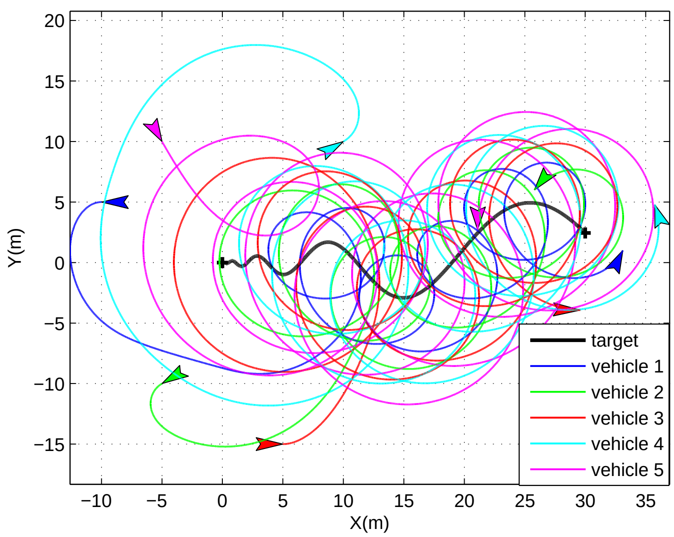

Fig. 3. The trajectories of each vehicle i, $\boldsymbol{p}_i,\ i=1,...,N$.

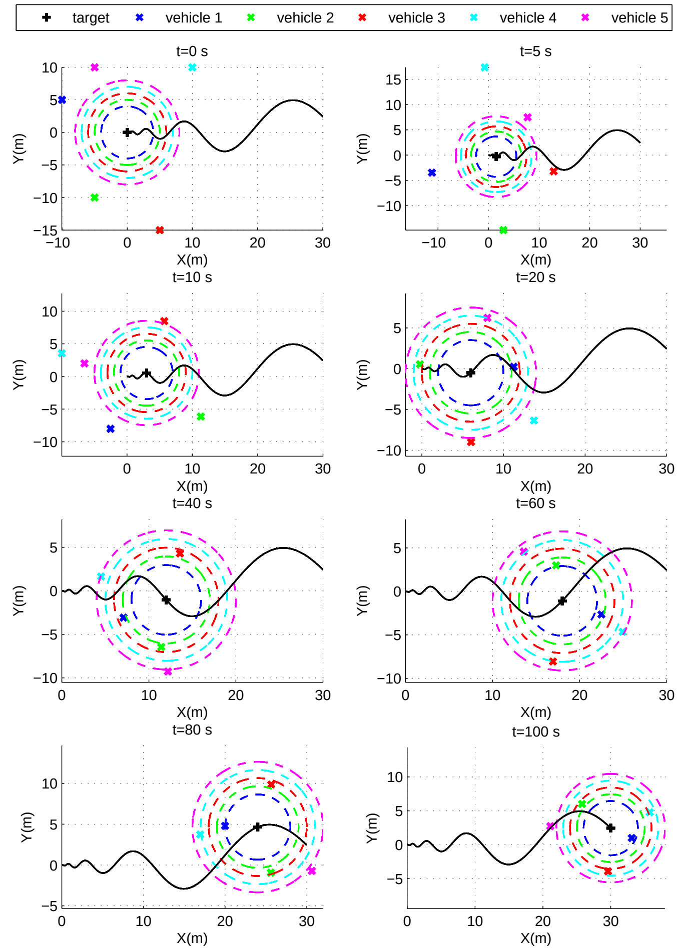

Fig. 4. The positions of each vehicle i, $\boldsymbol{p}_i,\ i=1,...,N$, at some instants.

$\boldsymbol{p}_3(t_0)=[5\ \ -15]^{\mathrm{T}},\ \theta_3(t_0)=0$, $\boldsymbol{p}_4(t_0)=[10\ 10]^{\mathrm{T}},\ \theta_4(t_0)=\frac{1}{6}\pi$, and $\boldsymbol{p}_5(t_0)=[-5\ 10]^{\mathrm{T}},\ \theta_5(t_0)=-\frac{1}{3}\pi$. The initial position of the target is given by $\boldsymbol{q}_0=[0\ 0]^{\mathrm{T}}$. The initial values of the estimates $\boldsymbol{\eta}_i(t_0)$ and $\dot{\boldsymbol{\eta}}_i(t_0)$, $i=1,...,N$, are given by $\boldsymbol{\eta}_1(t_0)=[2\ -6]^{\mathrm{T}}$, $\dot{\boldsymbol{\eta}}_1(t_0)=[0.3\ -0.25]^{\mathrm{T}}$, $\boldsymbol{\eta}_2(t_0)=[-3\ 8]^{\mathrm{T}}$, $\dot{\boldsymbol{\eta}}_2(t_0)=[0.35\ -0.6]^{\mathrm{T}}$, $\boldsymbol{\eta}_3(t_0)=[6\ -1]^{\mathrm{T}}$, $\dot{\boldsymbol{\eta}}_3(t_0)=[-0.25\ 0.5]^{\mathrm{T}}$, $\boldsymbol{\eta}_4(t_0)=[9\ 3]^{\mathrm{T}}$, $\dot{\boldsymbol{\eta}}_4(t_0)=[-0.5\ 0.1]^{\mathrm{T}}$, and $\boldsymbol{\eta}_5(t_0)=[-1\ \ -5]^{\mathrm{T}}$, $\dot{\boldsymbol{\eta}}_5(t_0)=[0.1\ 0.3]^{\mathrm{T}}$. Set $\mu=1$ and $\varepsilon_i=1$, and apply the proposed control law (9)–(10).

In this example, the target is moving along a trajectory $\boldsymbol{q}_0(t)=[0.3t\ \ 0.06t\sin(6\ln(t{+}1))]^{\mathrm{T}}$. It can be verified that $\dot{\boldsymbol{q}}_0$ and $\ddot{\boldsymbol{q}}_0$ satisfy Assumption 1. The trajectories of each vehicle and the target during 0–100s are shown in Fig. 3, and it is shown that all vehicles converge to a moving-target enclosing motion. Moreover, the positions of each vehicle i and the target at $t=\{0,5,10,20,40,60,80,100\}$s are shown in Fig. 4. Fig. 5 illustrates that all tracking errors $\|\boldsymbol{q}_i-\boldsymbol{q}_0\|$ converge to a small neighborhood of zero exponentially and are uniformly ultimately bounded. Fig. 6 shows that the angular velocity of each vehicle always satisfies the constraints (2).

Then, we show that by decreasing the ratio $\frac{r_i}{v_i}$, the ultimate bounds of the tracking errors $\|\boldsymbol{q}_i-\boldsymbol{q}_0\|$ can be smaller. Fig. 7 shows the tracking errors $\|\boldsymbol{q}_i-\boldsymbol{q}_0\|$ with smaller r_i, i.e., $r_1=2.5,\ r_2=3.125,\ r_3=3.75,\ r_4=4.375$, and $r_5=5$, and Fig. 8 shows the tracking errors $\|\boldsymbol{q}_i-\boldsymbol{q}_0\|$ with larger v_i, i.e., $v_1=3.2,\ v_2=4,\ v_3=4.8,\ v_4=5.6$, and $v_5=6.4$. It is obvious that the ultimate bounds in Figs. 7 and 8 are smaller than those in Fig. 5.

Finally, we consider the case where the target is stationary at $\boldsymbol{q}_0=[0\ 0]^{\mathrm{T}}$. The trajectories of each vehicle during 0–60s are shown in Fig. 9, and it is shown that all vehicles converge to enclosing the target. Fig. 10 validates that the tracking errors $\|\boldsymbol{q}_i-\boldsymbol{q}_0\|$ converge to 0. Fig. 11 illustrates that the angular velocity of each vehicle always satisfies the constraints (2).

The effectiveness of our proposed controllers is illustrated by these simulation results, and main result is also validated.

V. Conclusion

In this paper, we have studied the moving-target enclosing control problem of multiple unicycle-type vehicles with constant linear velocities. The topology of the network among the target and vehicles is described by a directed graph containing a spanning tree, which allows the information of the target is only known to some of the vehicles. A distributed dynamic control law has been developed, and it only requires each vehicle to use the information of itself and its neighbors. It has been shown that the tracking error with respect to the moving-target enclosing motion approaches zero with first-order approximation.

For the future work, we will focus on not only the implementation of the proposed control law for multiple mobile

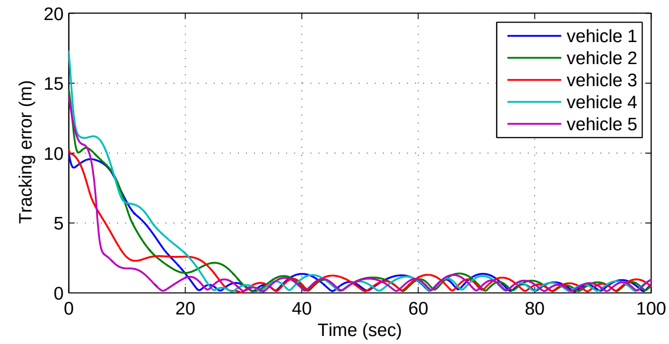

Fig. 5. The tracking error of each vehicle i, $\|\boldsymbol{q}_i - \boldsymbol{q}_0\|$, $i = 1, ..., N$.

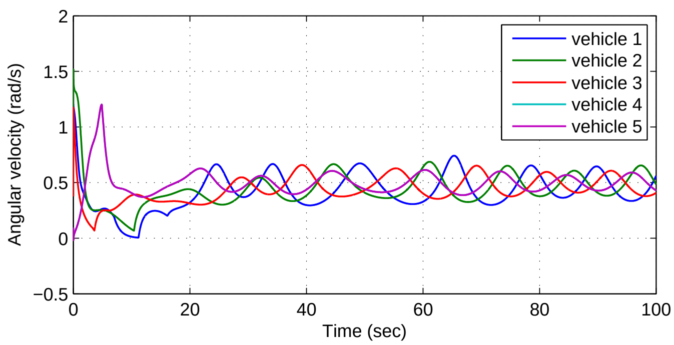

Fig. 6. The angular velocity of each vehicle i, ω_i, $i = 1, ..., N$.

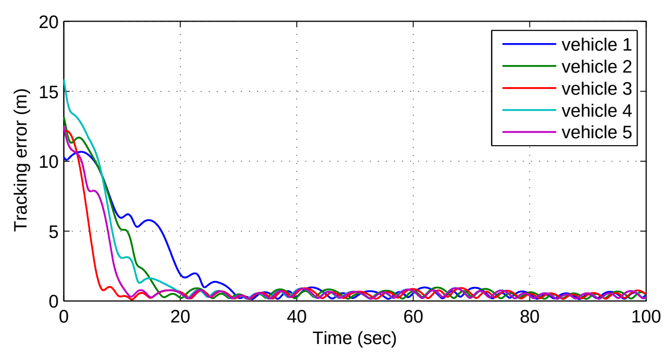

Fig. 7. The tracking error of each vehicle i, $\|\boldsymbol{q}_i - \boldsymbol{q}_0\|$, with smaller r_i.

wheeled robots, but also a new controller design which makes the tracking errors converge to zero under some certain assumptions. We will take into account the collision avoidance for networked vehicles in the same altitude. Besides, we will consider the case where the linear velocity is also a control input and will develop a controller for both linear and angular velocities to achieve both the circular motion and the position distribution around a moving target.

Appendix A
Proof of Proposition 3

For an instant $t \geq t_0$, $\forall t_0 \geq 0$, we have

$$\|\tilde{\boldsymbol{q}}_i(t)\| = \|\tilde{\boldsymbol{q}}_i(t_0)e^{-\boldsymbol{k}(\varepsilon_i)(t-t_0)} - \int_{t_0}^{t} e^{\boldsymbol{k}(\varepsilon_i)(\tau-t)} \times (\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i(\tau) + \dot{\tilde{\boldsymbol{\eta}}}_i(\tau))\mathrm{d}\tau\|$$

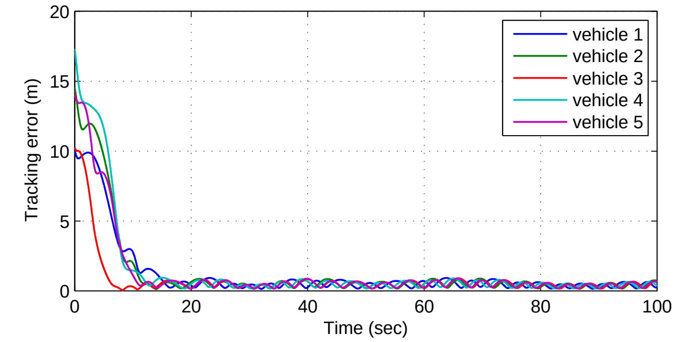

Fig. 8. The tracking error of each vehicle i, $\|\boldsymbol{q}_i - \boldsymbol{q}_0\|$, with larger v_i.

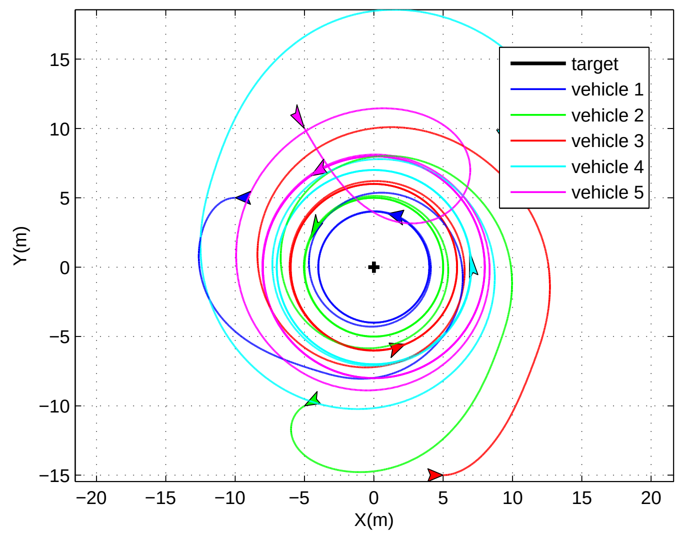

Fig. 9. The trajectories of each vehicle i, $\boldsymbol{p}_i$, $i = 1, ..., N$.

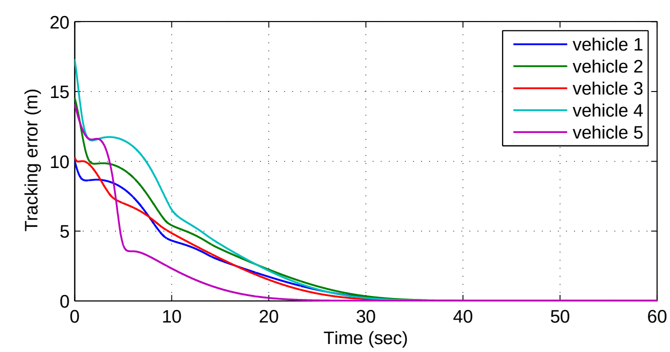

Fig. 10. The tracking error of each vehicle i, $\|\boldsymbol{q}_i - \boldsymbol{q}_0\|$, $i = 1, ..., N$.

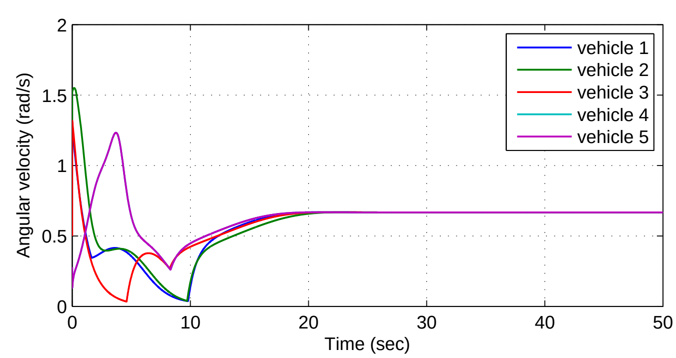

Fig. 11. The angular velocity of each vehicle i, ω_i, $i = 1, ..., N$.

$$
\begin{aligned}
&\leq \|\tilde{\boldsymbol{q}}_i(t_0)\| \|e^{-\boldsymbol{k}(\varepsilon_i)(t-t_0)}\| - \int_{t_0}^{t} \|e^{\boldsymbol{k}(\varepsilon_i)(\tau-t)}\| \\
&\quad \times (\|\boldsymbol{k}(\varepsilon_i)\tilde{\boldsymbol{\eta}}_i(\tau) + \dot{\tilde{\boldsymbol{\eta}}}_i(\tau)\|)\mathrm{d}\tau \\
&\leq \|\tilde{\boldsymbol{q}}_i(t_0)\| e^{-\varepsilon_i(t-t_0)} + e^{-\varepsilon_i t} \int_{t_0}^{t} e^{\varepsilon_i \tau} \\
&\quad \times (\varepsilon_i \|\tilde{\boldsymbol{\eta}}_i(\tau)\| + \|\dot{\tilde{\boldsymbol{\eta}}}_i(\tau)\|)\mathrm{d}\tau. \qquad (48)
\end{aligned}
$$

Since $[\boldsymbol{\eta}_i^{\mathrm{T}}(t)\ \ \dot{\boldsymbol{\eta}}_i^{\mathrm{T}}(t)]^{\mathrm{T}}$ converges to $[\boldsymbol{q}_0^{\mathrm{T}}(t)\ \ \boldsymbol{v}_0^{\mathrm{T}}(t)]^{\mathrm{T}}$ exponentially, there exist positive constants α, β, α', and β' such that $\|\tilde{\boldsymbol{\eta}}_i(t)\| \leq \alpha\|\tilde{\boldsymbol{\eta}}_i(t_0)\| e^{-\beta(t-t_0)}$ and $\|\dot{\tilde{\boldsymbol{\eta}}}_i(t)\| \leq \alpha'\|\dot{\tilde{\boldsymbol{\eta}}}_i(t_0)\| e^{-\beta'(t-t_0)}$, for all $[\tilde{\boldsymbol{\eta}}_i^{\mathrm{T}}(t_0)\ \ \dot{\tilde{\boldsymbol{\eta}}}_i^{\mathrm{T}}(t_0)]^{\mathrm{T}} \in \mathbb{R}^4$, $\forall t \geq t_0$. Then, (48) becomes

$$
\begin{aligned}
\|\tilde{\boldsymbol{q}}_i(t)\| &\leq \|\tilde{\boldsymbol{q}}_i(t_0)\| e^{-\varepsilon_i(t-t_0)} + \alpha\varepsilon_i e^{\beta t_0 - \varepsilon_i t} \int_{t_0}^{t} e^{(\varepsilon_i - \beta)\tau}\mathrm{d}\tau \\
&\quad + \alpha' e^{\beta' t_0 - \varepsilon_i t} \int_{t_0}^{t} e^{(\varepsilon_i - \beta')\tau}\mathrm{d}\tau. \qquad (49)
\end{aligned}
$$

Note that if $\varepsilon_i = \beta$, $e^{\beta t_0 - \varepsilon_i t}\int_{t_0}^{t} e^{(\varepsilon_i-\beta)\tau}\mathrm{d}\tau = e^{-\varepsilon_i(t-t_0)}(t - t_0)$, otherwise $e^{\beta t_0 - \varepsilon_i t}\int_{t_0}^{t} e^{(\varepsilon_i-\beta)\tau}\mathrm{d}\tau = \frac{e^{-\beta(t-t_0)} - e^{-\varepsilon_i(t-t_0)}}{\varepsilon_i - \beta}$. Thus, we can conclude from (49) that $\tilde{\boldsymbol{q}}_i(t)$ converges to $\mathbf{0}$ exponentially as $t \to \infty$.

The proof is thus completed. ■

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