Target Enclosing and Trajectory Tracking for a Mobile Robot With Input Disturbances

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***Abstract*—This letter investigates two control problems of a unicycle-type mobile robot: target enclosing and trajectory tracking. The existence of disturbance in velocities, i.e., the input channels of a mobile robot, is considered in both problems. The input disturbance is assumed to be generated by a linear exogenous system, which can be used to describe linear combinations of a finite number of sinusoidal signals and step signals. Two dynamic control laws are proposed respectively, such that the disturbance rejection is achieved and global asymptotic stability of the closed-loop system is guaranteed. Finally, simulation results of an example verify effectiveness of the proposed control laws.**

***Index Terms*—Adaptive control, autonomous vehicles, nonholonomic systems.**

I. Introduction

CONTROLLING a unicycle-type mobile robot has been a popular and important topic since the end of the last century, see a review [1] and references therein. Many works focus on two typical control problems of a mobile robot: target enclosing and trajectory tracking.

Enclosing a target by one or multiple mobile robots can be applied to securing, monitoring, or localizing an object of interest. Significant effort has been devoted to studying the target enclosing control problem in the past decade. In particular, a gradient control law was developed in [2], such that mobile robots move around a beacon with spacing arrangement. In [3], target enclosing of the mobile robot with limited visibility was investigated. In [4], a hybrid control law was developed based on some prescribed regions around the target. In [5], controllers without using bearing angle measurements were proposed for target enclosing.

Tracking a reference trajectory by a mobile robot is another fundamental and classic control problem which has been extensively studied. In [6], the problem was solved via a time-varying state feedback controller based on backstepping technique. In [7], sliding mode control technique was applied in controller design. In [8], velocity saturation was considered and the problem was solved if a P.E. condition on reference velocities is satisfied. In [9], a single saturated controller was proposed to simultaneously solve both tracking and regulation problems. In [10], the constraint of positive-minimum linear velocity was handled with the constrained control Lyapunov function method. The same velocity constraint was also resolved in [11] by properly using bounded functions in controller. In [12], an observer based controller was proposed in the absence of relative heading angle, and in [13], a vision based controller was developed without using relative position.

However, all aforementioned works did not take into account any disturbances, so that the obtained results were not applicable to many practical scenarios where there always exist some kinds of disturbances. In [14] and [15], bounded kinematic disturbances which violates the nonholonomic pure rolling and non-slipping constraint were considered in trajectory tracking control problem of a mobile robot. In [16], a comprehensive study on modeling and control of a mobile robot with the kinematic disturbances originated from the wheel skidding and slipping was presented. Later in [17], a GPS-based tracking controller was proposed to handle one of the disturbances in [16]. Some recent works considered the existence of disturbance in velocities, i.e., the input channels of a mobile robot. In [18], consensus of mobile robots with a bounded input disturbance generated by a linear exogenous system was achieved. The system matrix of the exogenous system satisfies a strong assumption, which limits the scope of the disturbances. In [19], the same disturbance was considered in distributed rendezvous and tracking control problems of mobile robots. In [20], the assumption on disturbances was relaxed to those subject to a known bound. Then, sliding mode control technique was used in controller design, which yields the chattering in the heading angles of mobile robots.

In this letter, a class of input disturbances generated by a linear exogenous system is considered. Two dynamic control laws are proposed for the target enclosing and trajectory tracking control problems respectively, such that both problems can be solved with full disturbance rejection. Different from [18] and [19], the system matrix of the exogenous system is only required to satisfy a mild assumption such that the concerned disturbances can describe linear combinations of a finite number of sinusoidal signals and step signals. For a mobile robot with such class of disturbances, our proposed control laws have advantages over that in [20] in the sense that the knowledge of the bound of the disturbances is not required, and no chattering will exist.

The rest of this letter is organized as follows. In Section II, we give the problem formulation. In Section III, two proposed control laws along with stability analysis are presented. Section IV shows simulation results of an example. In Section V, the conclusion is drawn.

Notations: For a matrix $A \in \mathbb{R}^{n\times n}$, $A > (<)\ \mathbf{0}$ and $A \geq (\leq)\ \mathbf{0}$ denote that A is positive (negative) definite and positive (negative) semi-definite respectively.

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II. Problem Formulation

Consider a unicycle-type mobile robot subject to disturbances in both linear velocity and angular velocity, and the kinematics the mobile robot is described by

$$\dot{x} = (v+\varepsilon_1)\cos\theta,\ \dot{y} = (v+\varepsilon_1)\sin\theta,\ \dot{\theta} = \omega+\varepsilon_2, \tag{1}$$

where $\boldsymbol{p} := [x\ y]^{\mathrm{T}} \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ are the position and heading angle of the robot in the inertial frame respectively. $v \in \mathbb{R}$ and $\omega \in \mathbb{R}$ are the linear velocity and angular velocity respectively and they are considered as the control inputs of system (1). $\boldsymbol{\varepsilon} := [\varepsilon_1\ \varepsilon_2]^{\mathrm{T}} \in \mathbb{R}^2$ is a disturbance generated by the following linear exogenous system

$$\boldsymbol{\varepsilon} = [\boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{z}\ \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{z}]^{\mathrm{T}}, \dot{\boldsymbol{z}} = S\boldsymbol{z}, \tag{2}$$

where $\boldsymbol{z} \in \mathbb{R}^m$, $\boldsymbol{b}_1$, $\boldsymbol{b}_2 \in \mathbb{R}^m$ are constant vectors, and the constant matrix $S \in \mathbb{R}^{m\times m}$ satisfies the following assumption.

[A1] The matrix S is marginally stable, i.e., the eigenvalues of S have non-positive real part and those eigenvalues with zero real part are semi-simple.

In this letter, we consider two control problems of a mobile robot (1) with respect to a target located at $\boldsymbol{p}_0 := [x_0\ y_0]^{\mathrm{T}}$.

First, the *target enclosing* control problem is studied. The objective is to design $[v\ \omega]^{\mathrm{T}}$ such that a mobile robot (1) can encircle a given stationary target $\boldsymbol{p}_0$ with a counterclockwise circular motion.

Second, the *trajectory tracking* control problem is investigated. The objective is to design $[v\ \omega]^{\mathrm{T}}$ such that a mobile robot (1) can track a reference trajectory generated by a moving target with the following dynamics:

$$\dot{x}_0 = v_0\cos\theta_0,\ \dot{y}_0 = v_0\sin\theta_0,\ \dot{\theta}_0 = \omega_0, \tag{3}$$

where v_0 and ω_0 satisfy the following assumption:

[A2] $v_0(t)$ and $\omega_0(t)$ are differentiable in t, and $\dot{v}_0(t)$ and $\dot{\omega}_0(t)$ are bounded.

The mobile robot (1) is allowed to use its local coordinate frame, i.e., the Frenet-Serret frame, with the origin at its position $\boldsymbol{p}$ and the x-axis coincident with its orientation θ. Denote the error coordinate with respect to the target by using the following coordinate transformation [21]:

$$\boldsymbol{p}_e = [x_e\ y_e]^{\mathrm{T}} = R(\theta)(\boldsymbol{p}_0 - \boldsymbol{p}),\ \theta_e = \theta_0 - \theta. \tag{4}$$

where $R(\cdot) = \begin{bmatrix} \cos(\cdot) & \sin(\cdot) \\ -\sin(\cdot) & \cos(\cdot) \end{bmatrix}$.

Then, based on the definition of the problem of *simultaneous stabilization and tracking* [1, Definition 1], the *target enclosing* and *trajectory tracking* control problems considered in this letter are formally defined as follows.

*Problem 1 (***Target enclosing control problem):** Given a target located at $\boldsymbol{p}_0 = [x_0\ y_0]^{\mathrm{T}}$ and a desired radius r, for a mobile robot (1) subject to a disturbance generated by system (2), with any initial states $[\boldsymbol{p}^{\mathrm{T}}(t_0)\ \theta(t_0)]^{\mathrm{T}} \in \mathbb{R}^3$, $\forall t_0 \geq 0$, find a dynamic control law in the form of

$$[v\ \omega]^{\mathrm{T}} = \boldsymbol{\sigma}(\boldsymbol{p}_e, \boldsymbol{\rho}, r),\ \dot{\boldsymbol{\rho}} = \boldsymbol{\varsigma}(\boldsymbol{p}_e, \boldsymbol{\rho}, r), \tag{5}$$

such that $\boldsymbol{p}(t)$ is bounded for all $t \geq t_0$ and

$$\lim_{t\to\infty}(\boldsymbol{p}(t) - \boldsymbol{p}_0) = r\left[\sin\theta(t)\ \ -\cos\theta(t)\right]^{\mathrm{T}}. \tag{6}$$

where $\boldsymbol{\rho}$, to be designed later, is an internal state, and $\boldsymbol{\sigma}(\cdot)$ and $\boldsymbol{\varsigma}(\cdot)$ are sufficiently smooth functions. ■

*Problem 2 (***Trajectory tracking control problem):** Given a reference trajectory $[\boldsymbol{p}_0^{\mathrm{T}}\ \theta_0]^{\mathrm{T}}$ generated by system (3), for a mobile robot (1) subject to a disturbance generated by system (2), with any initial states $[\boldsymbol{p}^{\mathrm{T}}(t_0)\ \theta(t_0)]^{\mathrm{T}} \in \mathbb{R}^3$, $\forall t_0 \geq 0$, find a dynamic control law in the form of

$$[v\ \omega]^{\mathrm{T}} = \boldsymbol{\sigma}(\boldsymbol{p}_e, \theta_e, \boldsymbol{\rho}, v_0, \omega_0),\ \dot{\boldsymbol{\rho}} = \boldsymbol{\varsigma}(\boldsymbol{p}_e, \theta_e, \boldsymbol{\rho}, v_0, \omega_0), \tag{7}$$

such that $\boldsymbol{p}(t) - \boldsymbol{p}_0(t)$ is bounded for all $t \geq t_0$ and

$$\lim_{t\to\infty}(\boldsymbol{p}(t) - \boldsymbol{p}_0(t)) = \boldsymbol{0}, \lim_{t\to\infty}(\theta(t) - \theta_0(t)) = 2K\pi. \tag{8}$$

where K is some integer, $\boldsymbol{\rho}$ is an internal state to be designed, and $\boldsymbol{\sigma}(\cdot)$ and $\boldsymbol{\varsigma}(\cdot)$ are sufficiently smooth functions. ■

Remark 2.1: Under assumption [A1], $\boldsymbol{\varepsilon} = [\varepsilon_1\ \varepsilon_2]^{\mathrm{T}}$ can describe linear combinations of a finite number of sinusoidal signals and step signals, and can be written in the form of

$$\varepsilon_i = \alpha_i + \textstyle\sum_{j=1}^{n_i}\beta_{ij}\sin(\gamma_{ij}t + \phi_{ij}), i = 1, 2, \tag{9}$$

with unknown α_i, β_{ij}, and ϕ_{ij}, $j = 1, ..., n_i$. It is known that a periodic signal can be represented as a sum of sinusoids by Fourier series expansion. Thus, a large class of persistent external disturbances with unbounded energy are included in the problem formulation. In practice, some input disturbances can be persistent and periodical, which is caused by some malfunction of engine or actuation, such as motor offset and wear and tear of devices, and by some external environmental perturbations, such as steady wind, airflow, and friction. ■

Remark 2.2: In [18] and [19], the disturbance $\boldsymbol{\varepsilon} = [\varepsilon_1\ \varepsilon_2]^{\mathrm{T}}$ was generated by the following linear exogenous system

$$\varepsilon_1 = \boldsymbol{b}_1^{\mathrm{T}}\boldsymbol{z}_1,\ \dot{\boldsymbol{z}}_1 = S_1\boldsymbol{z}_1,\ \varepsilon_2 = \boldsymbol{b}_2^{\mathrm{T}}\boldsymbol{z}_2,\ \dot{\boldsymbol{z}}_2 = S_2\boldsymbol{z}_2. \tag{10}$$

Note that system (2) can also be written in the form of (10), and vice versa. Vectors $\boldsymbol{b}_1, \boldsymbol{b}_2$, and matrices S_1 and S_2 were assumed to satisfy [A1'] $\boldsymbol{b}_1\boldsymbol{b}_1^{\mathrm{T}}S_1$ and $\boldsymbol{b}_2\boldsymbol{b}_2^{\mathrm{T}}S_2$ are negative semi-definite [18], or [A1''] $\boldsymbol{b}_1\boldsymbol{b}_1^{\mathrm{T}}S_1 + S_1^{\mathrm{T}}\boldsymbol{b}_1\boldsymbol{b}_1^{\mathrm{T}}$ and $\boldsymbol{b}_2\boldsymbol{b}_2^{\mathrm{T}}S_2 + S_2^{\mathrm{T}}\boldsymbol{b}_2\boldsymbol{b}_2^{\mathrm{T}}$ are negative semi-definite [19]. These assumptions imply that $\boldsymbol{\varepsilon}$ will not diverge, which can be observed by choosing a Lyapunov function candidate $V_{\varepsilon_1} = \frac{1}{2}\varepsilon_1^2$ and then obtaining

$$\dot{V}_{\varepsilon_1} = \boldsymbol{z}_1^{\mathrm{T}}b_1b_1^{\mathrm{T}}S_1\boldsymbol{z}_1 = \tfrac{1}{2}\boldsymbol{z}_1^{\mathrm{T}}(\boldsymbol{b}_1\boldsymbol{b}_1^{\mathrm{T}}S_1 + S_1^{\mathrm{T}}\boldsymbol{b}_1\boldsymbol{b}_1^{\mathrm{T}})\boldsymbol{z}_1 \leq 0.$$

However, [A1'] and [A1''] limit the scope of the disturbances in the sense that $\boldsymbol{\varepsilon}(t)$ cannot represent linear combinations of a finite number of sinusoidal signals. For example, $\boldsymbol{\varepsilon}(t) = [\frac{\sqrt{2}}{6} + \frac{1}{3}\sin(\frac{3}{5}t + \frac{\pi}{4})\ \ \frac{\sqrt{2}}{6} + \frac{1}{15}\sin(\frac{3}{5}t + \frac{\pi}{4}) - \frac{1}{5}\cos(\frac{3}{5}t + \frac{\pi}{4})]^{\mathrm{T}}$ is a signal generated by the exogenous system (10) with $S_1 = S_2 = \begin{bmatrix} 0.2 & -1 & 0 \\ 0.4 & -0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\boldsymbol{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. It can be verified that assumptions [A1'] and [A1''] are not satisfied. While this $\boldsymbol{\varepsilon}(t)$ is a signal in the form of (9) which is generated by system (2) with $S = S_1$ satisfying [A1]. Note that there exists a matrix $P > \boldsymbol{0}$ satisfying

$$PS + S^{\mathrm{T}}P \leq \boldsymbol{0}, \tag{11}$$

if and only if assumption [A1] is satisfied. ■

III. Main Results

In this section, we present solutions to the *target enclosing* and *trajectory tracking* control problems along with stability analysis of the closed-loop systems respectively.

A. Solution to Target Enclosing Control Problem

First, we solve the *target enclosing* control problem.

Define a tracking error $\boldsymbol{e} := [e_x \ e_y]^{\mathrm{T}}$ as

$$\boldsymbol{e} := [e_x \ e_y]^{\mathrm{T}} = \boldsymbol{p}_e - \begin{bmatrix} 0 \\ r \end{bmatrix}. \tag{12}$$

Then, the error dynamics can be expressed as

$$\dot{\boldsymbol{e}} = (\omega + \varepsilon_2) A\boldsymbol{e} - (v - \omega r + \varepsilon_1 - \varepsilon_2 r) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{13}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. To achieve (6), it suffices to show

$$\lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{0}. \tag{14}$$

To find a solution in the form of (5), we first introduce the following internal states:

$$\hat{\boldsymbol{e}} := [\hat{e}_x \ \hat{e}_y]^{\mathrm{T}} \in \mathbb{R}^2, \ \hat{\boldsymbol{z}} \in \mathbb{R}^m, \ \hat{\boldsymbol{\varepsilon}} := [\hat{\varepsilon}_1 \ \hat{\varepsilon}_2]^{\mathrm{T}} = [\boldsymbol{b}_1^{\mathrm{T}}\hat{\boldsymbol{z}} \ \boldsymbol{b}_2^{\mathrm{T}}\hat{\boldsymbol{z}}]^{\mathrm{T}}.$$

Then, the following dynamic control law is proposed.

$$v = v_0 - \hat{\varepsilon}_1, \tag{15}$$

$$\omega = \frac{v_0}{r} - \hat{\varepsilon}_2 - \frac{k}{r}\frac{\hat{e}_x}{\sqrt{1+\hat{e}_x^2}}, \tag{16}$$

$$\dot{\hat{\boldsymbol{e}}} = (\omega + \hat{\varepsilon}_2) A\hat{\boldsymbol{e}} + L(\boldsymbol{e} - \hat{\boldsymbol{e}}) - \frac{k(\hat{e}_x - e_x)}{\sqrt{1+\hat{e}_x^2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{17}$$

$$\dot{\hat{\boldsymbol{z}}} = S\hat{\boldsymbol{z}} + P^{-1}(\boldsymbol{b}_2(\boldsymbol{e} - \hat{\boldsymbol{e}})^{\mathrm{T}} A\boldsymbol{e} - [\boldsymbol{b}_1 \ \boldsymbol{b}_2] \begin{bmatrix} -1 \\ r \end{bmatrix} (\hat{e}_x - 2e_x)), \tag{18}$$

where v_0 is a non-zero constant, k is any positive constant, $L \in \mathbb{R}^{2\times 2}$ is any matrix satisfying $L + L^{\mathrm{T}} > \boldsymbol{0}$, and $P \in \mathbb{R}^{m\times m}$ is a positive definite matrix satisfying inequality (11) under assumption [A1]. The initial states $\hat{\boldsymbol{e}}(t_0)$ and $\hat{\boldsymbol{z}}(t_0)$ can be arbitrarily chosen in $\mathbb{R}^2$ and $\mathbb{R}^m$ respectively.

Then, we have the following theorem.

Theorem 1: The *target enclosing* control problem, i.e., Problem 1, is solved by control law (15)–(18) under assumption [A1]. ∎

Proof: Define $\tilde{\boldsymbol{e}} := [\tilde{e}_x \ \tilde{e}_y]^{\mathrm{T}}$, $\tilde{\boldsymbol{z}}$, and $\tilde{\boldsymbol{\varepsilon}} := [\tilde{\varepsilon}_1 \ \tilde{\varepsilon}_2]^{\mathrm{T}}$ as

$$\tilde{\boldsymbol{e}} = \hat{\boldsymbol{e}} - \boldsymbol{e}, \ \tilde{\boldsymbol{z}} = \hat{\boldsymbol{z}} - \boldsymbol{z}, \ \tilde{\boldsymbol{\varepsilon}} = \hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}. \tag{19}$$

The augmented closed-loop system consisting of (2), (13), and (15)–(18) can be written as

$$\dot{\boldsymbol{e}} = (\omega + \varepsilon_2) A\boldsymbol{e} - (\tilde{\varepsilon}_2 r - \tilde{\varepsilon}_1 + \frac{k(\tilde{e}_x + e_x)}{\sqrt{1+\hat{e}_x^2}}) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{20}$$

$$\begin{aligned} \dot{\tilde{\boldsymbol{e}}} &= (\omega + \hat{\varepsilon}_2) A\tilde{\boldsymbol{e}} + \tilde{\varepsilon}_2 A\boldsymbol{e} - L\tilde{\boldsymbol{e}} \\ &\quad + (\tilde{\varepsilon}_2 r - \tilde{\varepsilon}_1 + \frac{k e_x}{\sqrt{1+\hat{e}_x^2}}) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned} \tag{21}$$

$$\dot{\tilde{\boldsymbol{z}}} = S\tilde{\boldsymbol{z}} - P^{-1}(\boldsymbol{b}_2 \tilde{\boldsymbol{e}}^{\mathrm{T}} A\boldsymbol{e} + [\boldsymbol{b}_1 \ \boldsymbol{b}_2] \begin{bmatrix} -1 \\ r \end{bmatrix} (\tilde{e}_x - e_x)). \tag{22}$$

Consider a Lyapunov function candidate $V(t, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}$ as

$$V(t, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}}) = \tfrac{1}{2}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} + \tfrac{1}{2}\tilde{\boldsymbol{e}}^{\mathrm{T}}\tilde{\boldsymbol{e}} + \tfrac{1}{2}\tilde{\boldsymbol{z}}^{\mathrm{T}}P\tilde{\boldsymbol{z}}, \tag{23}$$

which is positive definite and decrescent. Taking the time derivative of $V(t, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}})$ along the trajectories of system (20)–(22) yields

$$\begin{aligned} \dot{V} &= \frac{\omega + \varepsilon_2}{2}\boldsymbol{e}^{\mathrm{T}}(A + A^{\mathrm{T}})\boldsymbol{e} - e_x(\tilde{\varepsilon}_2 r - \tilde{\varepsilon}_1) - \frac{k(e_x\tilde{e}_x + e_x^2)}{\sqrt{1+\hat{e}_x^2}} \\ &\quad + \frac{\omega + \hat{\varepsilon}_2}{2}\tilde{\boldsymbol{e}}^{\mathrm{T}}(A + A^{\mathrm{T}})\tilde{\boldsymbol{e}} + \tilde{\varepsilon}_2\tilde{\boldsymbol{e}}^{\mathrm{T}}A\boldsymbol{e} - \tfrac{1}{2}\tilde{\boldsymbol{e}}^{\mathrm{T}}(L + L^{\mathrm{T}})\tilde{\boldsymbol{e}} \\ &\quad + \tilde{e}_x(\tilde{\varepsilon}_2 r - \tilde{\varepsilon}_1) + \frac{k e_x\tilde{e}_x}{\sqrt{1+\hat{e}_x^2}} + \tfrac{1}{2}\tilde{\boldsymbol{z}}^{\mathrm{T}}(PS + S^{\mathrm{T}}P)\tilde{\boldsymbol{z}} \\ &\quad - \tilde{\varepsilon}_2\tilde{\boldsymbol{e}}^{\mathrm{T}}A\boldsymbol{e} - (\tilde{\varepsilon}_2 r - \tilde{\varepsilon}_1)(\tilde{e}_x - e_x) \\ &\leq -\frac{k e_x^2}{\sqrt{1+\hat{e}_x^2}} - \tfrac{1}{2}\tilde{\boldsymbol{e}}^{\mathrm{T}}(L + L^{\mathrm{T}})\tilde{\boldsymbol{e}} \leq 0, \end{aligned} \tag{24}$$

where it is noted that $A + A^{\mathrm{T}} = \boldsymbol{0}$, $PS + S^{\mathrm{T}}P \leq \boldsymbol{0}$ and $L + L^{\mathrm{T}} > \boldsymbol{0}$. $\dot{V}(t, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}}) \leq 0$ shows that the augmented closed-loop system (20)–(22) is uniformly stable, and that $V(t, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}})$ is nonincreasing in t and bounded. Then, $\lim_{t\to\infty}\int_{t_0}^{t}\dot{V}(\tau, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}})\mathrm{d}\tau$ exists and is finite. It follows from $V(t, \boldsymbol{e}(t), \tilde{\boldsymbol{e}}(t), \tilde{\boldsymbol{z}}(t)) \leq V(t_0, \boldsymbol{e}(t_0), \tilde{\boldsymbol{e}}(t_0), \tilde{\boldsymbol{z}}(t_0))$ that $\boldsymbol{e}$, $\tilde{\boldsymbol{e}}$, and $\tilde{\boldsymbol{z}}$ are bounded, which implies that $\boldsymbol{p}$, $\tilde{\boldsymbol{\varepsilon}}$, $\dot{\boldsymbol{e}}$, $\dot{\tilde{\boldsymbol{e}}}$, and $\dot{\tilde{\boldsymbol{z}}}$ are bounded. Hence, $\ddot{V}(t, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}})$ is bounded and $\dot{V}(t, \boldsymbol{e}, \tilde{\boldsymbol{e}}, \tilde{\boldsymbol{z}})$ is uniformly continuous in t. By Barbalat's Lemma,

$$\lim_{t\to\infty} e_x(t) = 0, \ \lim_{t\to\infty} \tilde{\boldsymbol{e}}(t) = \boldsymbol{0}. \tag{25}$$

Next, we employ the extended Barbalat's Lemma (Lemma A.1) to show $\lim_{t\to\infty} e_y(t) = 0$.

It follows from (17) that the time derivative of $\hat{e}_x$ is

$$\dot{\hat{e}}_x = (\omega + \hat{\varepsilon}_2)(e_y + \tilde{e}_y) - \frac{k\tilde{e}_x}{\sqrt{1+\hat{e}_x^2}} - [1 \ \ 0]L\tilde{\boldsymbol{e}}. \tag{26}$$

Then, let

$$h_1 = (\omega + \hat{\varepsilon}_2)e_y = \frac{v_0}{r}e_y, \tag{27}$$

$$h_2 = (\omega + \hat{\varepsilon}_2)\tilde{e}_y - \frac{k\tilde{e}_x}{\sqrt{1+\hat{e}_x^2}} - [1 \ \ 0]L\tilde{\boldsymbol{e}}. \tag{28}$$

It follows from (25) and (28) that $\lim_{t\to\infty} h_2(t) = 0$. Note that $\boldsymbol{e}$ is bounded and $\boldsymbol{\varepsilon}$ is also bounded under assumption [A1]. Then, it follows from (20) and (27) that $\dot{h}_1(t)$ exists and is bounded, and then $h_1(t)$ is uniformly continuous in t. By Lemma A.1, $\lim_{t\to\infty} h_1(t) = 0$, and thus $\lim_{t\to\infty} e_y(t) = 0$.

This completes the proof. ∎

Remark 3.1: Introducing the internal state $\hat{\boldsymbol{e}}$ along with (17) helps establish the result $\lim_{t\to\infty} e_y(t) = 0$. Introducing $\hat{\boldsymbol{\varepsilon}}$ only and using an adaptive control law in the form of

$$[v \ \omega]^{\mathrm{T}} = \boldsymbol{\sigma}(\boldsymbol{e}, \hat{\boldsymbol{z}}, r), \ \dot{\hat{\boldsymbol{z}}} = \boldsymbol{\varsigma}(\boldsymbol{e}, \hat{\boldsymbol{z}}, r), \tag{29}$$

would lead to the lack of a suitable differentiable function for proving $\lim_{t\to\infty} e_y(t) = 0$ by the extended Barbalat's Lemma. Then, only $\lim_{t\to\infty} e_x(t) = 0$ can be shown and $\lim_{t\to\infty} e_y(t) = 0$ cannot be guaranteed. ∎

Remark 3.2: In the case where v is required to be a constant [3], i.e., $v = v_0$, ω in (16) can be modified as

$$\omega = \frac{v_0}{r} - \hat{\varepsilon}_2 + \frac{\hat{\varepsilon}_1}{r} - \frac{k}{r}\frac{\hat{e}_x}{\sqrt{1+\hat{e}_x^2}}. \tag{30}$$

Moreover, if $S = \mathbf{0}$, the constant v_0 needs to be adjusted if (1) $v_0 + \varepsilon_1(t) = 0$ and (2) $v_0 + \lim_{t\to\infty}\hat{\varepsilon}_1(t) = 0$. The first case is trivial since the mobile robot cannot move if $\dot{\boldsymbol{p}} \equiv \mathbf{0}$. The second case occurs when $\hat{\varepsilon}_1(t)$ happens to converge to $-v_0$. With $v = v_0$ and ω in (30), (27) becomes $h_1 = \frac{1}{r}(v_0 + \hat{\varepsilon}_1)e_y$. In this case, v_0 is required to be adjusted such that $\lim_{t\to\infty} e_y(t) = 0$ can be guaranteed. ■

Remark 3.3: The target enclosing control problem of one or multiple mobile robots was studied by several works with different focuses, see [2]–[5]. However, these works did not consider any disturbances. While the existence of disturbance in velocities, i.e., the input channels of the mobile robot is taken into account in this paper. ■

B. Solution to Trajectory Tracking Control Problem

Then, we solve the *trajectory tracking* control problem.

Use $[\boldsymbol{p}_e^{\mathrm{T}}\ \theta_e]^{\mathrm{T}}$ as the tracking error, and the error dynamics can be expressed as

$$\dot{\boldsymbol{p}}_e = (\omega+\varepsilon_2)A\boldsymbol{p}_e - \begin{bmatrix} v+\varepsilon_1 \\ 0 \end{bmatrix} + v_0\begin{bmatrix} \cos\theta_e \\ \sin\theta_e \end{bmatrix}, \tag{31}$$

$$\dot{\theta}_e = \omega_0 - \omega - \varepsilon_2. \tag{32}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. To achieve (8), it suffices to show

$$\lim_{t\to\infty}\boldsymbol{p}_e(t) = \mathbf{0},\ \ \lim_{t\to\infty}\sin\tfrac{\theta_e(t)}{2} = 0. \tag{33}$$

To find a solution in the form of (7), we introduce the following internal states:

$$\hat{\boldsymbol{p}}_e := [\hat{x}_e\ \hat{y}_e]^{\mathrm{T}} \in \mathbb{R}^2,\ \hat{\boldsymbol{z}} \in \mathbb{R}^m,\ \hat{\boldsymbol{\varepsilon}} := [\hat{\varepsilon}_1\ \hat{\varepsilon}_2]^{\mathrm{T}} = [\boldsymbol{b}_1^{\mathrm{T}}\hat{\boldsymbol{z}}\ \boldsymbol{b}_2^{\mathrm{T}}\hat{\boldsymbol{z}}]^{\mathrm{T}},$$

and propose the following dynamic control law:

$$v = v_0 + \frac{k_1\hat{x}_e}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} - \hat{\varepsilon}_1, \tag{34}$$

$$\omega = \omega_0 + \frac{k_2 v_0[-\sin\frac{\theta_e}{2}\ \ \cos\frac{\theta_e}{2}]\boldsymbol{p}_e + k_3\sin\frac{\theta_e}{2}}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} - \hat{\varepsilon}_2, \tag{35}$$

$$\begin{aligned}\dot{\hat{\boldsymbol{p}}}_e &= (\omega+\hat{\varepsilon}_2)A\hat{\boldsymbol{p}}_e - \begin{bmatrix} v+\hat{\varepsilon}_1 \\ 0 \end{bmatrix} + v_0\begin{bmatrix} \cos\theta_e \\ \sin\theta_e \end{bmatrix} \\ &\quad + \frac{k_2 v_0[-1+\cos\theta_e\ \ \sin\theta_e]\boldsymbol{p}_e(\boldsymbol{p}_e+\hat{\boldsymbol{p}}_e)}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}(\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}+\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e})} \\ &\quad + \frac{k_1k_2x_e}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + L(\boldsymbol{p}_e - \hat{\boldsymbol{p}}_e),\end{aligned} \tag{36}$$

$$\begin{aligned}\dot{\hat{\boldsymbol{z}}} &= S\hat{\boldsymbol{z}} + P^{-1}b_2((\boldsymbol{p}_e-\hat{\boldsymbol{p}}_e)^{\mathrm{T}}A\boldsymbol{p}_e - 2\sin\tfrac{\theta_e}{2}) \\ &\quad - P^{-1}b_1(x_e - \hat{x}_e + \frac{k_2x_e}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}}),\end{aligned} \tag{37}$$

where k_1 and k_3 are any positive constants, k_2 satisfies $0 < k_2 \le |\lim_{t\to\infty}\omega_0(t)/\lim_{t\to\infty}v_0(t)|$ if both $v_0(t)$ and $\omega_0(t)$ have finite and non-zero limits as $t\to\infty$; otherwise k_2 is any positive constant, $L \in \mathbb{R}^{2\times 2}$ is any matrix satisfying $L + L^{\mathrm{T}} > \mathbf{0}$, $P \in \mathbb{R}^{m\times m}$ is a positive definite matrix satisfying inequality (11) under assumption [A1]. The initial states $[\hat{\boldsymbol{p}}_e^{\mathrm{T}}(t_0)\ \hat{\boldsymbol{z}}^{\mathrm{T}}(t_0)]^{\mathrm{T}}$ can be arbitrarily chosen in $\mathbb{R}^2\times\mathbb{R}^m$.

Then, we have the following theorem.

Theorem 2: The *trajectory tracking* control problem, i.e., Problem 2, is solved by control law (34)–(37) if assumptions [A1]–[A2] are satisfied and either $v_0(t)$ or $\omega_0(t)$ does not converge to zero as $t\to\infty$. ■

Proof: Define $\tilde{\boldsymbol{p}}_e := [\tilde{x}_e\ \tilde{y}_e]^{\mathrm{T}}$, $\tilde{\boldsymbol{z}}$, and $\tilde{\boldsymbol{\varepsilon}} := [\tilde{\varepsilon}_1\ \tilde{\varepsilon}_2]^{\mathrm{T}}$ as

$$\tilde{\boldsymbol{p}}_e = \hat{\boldsymbol{p}}_e - \boldsymbol{p}_e,\ \tilde{\boldsymbol{z}} = \hat{\boldsymbol{z}} - \boldsymbol{z},\ \tilde{\boldsymbol{\varepsilon}} = \hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}. \tag{38}$$

The augmented closed-loop system consisting of (2), (31), and (34)–(37) can be written as

$$\dot{\boldsymbol{p}}_e = (\omega+\varepsilon_2)A\boldsymbol{p}_e + \begin{bmatrix} v_0(\cos\theta_e-1) - \frac{k_1(\tilde{x}_e+x_e)}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} + \tilde{\varepsilon}_1 \\ v_0\sin\theta_e \end{bmatrix}, \tag{39}$$

$$\dot{\theta}_e = \frac{k_2v_0(x_e\sin\frac{\theta_e}{2} - y_e\cos\frac{\theta_e}{2}) - k_3\sin\frac{\theta_e}{2}}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} + \tilde{\varepsilon}_2, \tag{40}$$

$$\begin{aligned}\dot{\tilde{\boldsymbol{p}}}_e &= (\omega+\hat{\varepsilon}_2)A\tilde{\boldsymbol{p}}_e + \tilde{\varepsilon}_2A\boldsymbol{p}_e - L\tilde{\boldsymbol{p}}_e \\ &\quad + \frac{k_2v_0(x_e(\cos\theta_e-1)+y_e\sin\theta_e)(\boldsymbol{p}_e+\hat{\boldsymbol{p}}_e)}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}(\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}+\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e})} \\ &\quad + (\frac{k_1k_2x_e}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}} - \tilde{\varepsilon}_1)\begin{bmatrix} 1 \\ 0 \end{bmatrix},\end{aligned} \tag{41}$$

$$\begin{aligned}\dot{\tilde{\boldsymbol{z}}} &= S\tilde{\boldsymbol{z}} - P^{-1}b_2(\tilde{\boldsymbol{p}}_e^{\mathrm{T}}A\boldsymbol{p}_e + 2\sin\tfrac{\theta_e}{2}) \\ &\quad + P^{-1}b_1(\tilde{x}_e - \frac{k_2x_e}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}}).\end{aligned} \tag{42}$$

Consider a Lyapunov function candidate $V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}}) : \mathbb{R}_{\ge 0}\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^m \to \mathbb{R}$ as

$$\begin{aligned}V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}}) &= \tfrac{1}{2}\tilde{\boldsymbol{p}}_e^{\mathrm{T}}\tilde{\boldsymbol{p}}_e + \tfrac{1}{2}\tilde{\boldsymbol{z}}^{\mathrm{T}}P\tilde{\boldsymbol{z}} + 8\sin^2\tfrac{\theta_e}{4} \\ &\quad + k_2(\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e} - 1),\end{aligned} \tag{43}$$

which is positive definite and decrescent. Taking the time derivative of $V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}})$ along the trajectories of system (39)–(42) yields

$$\begin{aligned}\dot{V} &= -\frac{k_1k_2x_e(x_e+\tilde{x}_e)}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} + \frac{k_2x_e\tilde{\varepsilon}_1}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}} - \frac{2k_3\sin^2\frac{\theta_e}{2}}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} \\ &\quad + 2\tilde{\varepsilon}_2\sin\tfrac{\theta_e}{2} + \frac{k_2v_0(x_e(\cos\theta_e-1)+y_e\sin\theta_e)}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} \\ &\quad - \frac{2k_2v_0\sin\frac{\theta_e}{2}(y_e\cos\frac{\theta_e}{2} - x_e\sin\frac{\theta_e}{2})}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}} + \tilde{\varepsilon}_2\tilde{\boldsymbol{p}}_eA\boldsymbol{p}_e \\ &\quad - \tilde{\varepsilon}_1\tilde{x}_e + \tfrac{1}{2}(\omega+\hat{\varepsilon}_2)\tilde{\boldsymbol{p}}_e^{\mathrm{T}}(A+A^{\mathrm{T}})\tilde{\boldsymbol{p}}_e - \tfrac{1}{2}\tilde{\boldsymbol{p}}_e^{\mathrm{T}}(L+L^{\mathrm{T}})\tilde{\boldsymbol{p}}_e \\ &\quad + \frac{k_1k_2x_e\tilde{x}_e}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} - k_2v_0(x_e(\cos\theta_e-1) \\ &\quad + y_e\sin\theta_e)(\frac{1}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} - \frac{1}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}}) - 2\tilde{\varepsilon}_2\sin\tfrac{\theta_e}{2} \\ &\quad - \tilde{\varepsilon}_2\tilde{\boldsymbol{p}}_eA\boldsymbol{p}_e + \tilde{\varepsilon}_1\tilde{x}_e - \frac{k_2\tilde{\varepsilon}_1x_e}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}} + \tfrac{1}{2}\tilde{\boldsymbol{z}}^{\mathrm{T}}(PS+S^{\mathrm{T}}P)\tilde{\boldsymbol{z}} \\ &= \frac{-k_1k_2x_e^2}{\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} - \frac{2k_3\sin^2\frac{\theta_e}{2}}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}} - \tfrac{1}{2}\tilde{\boldsymbol{p}}_e^{\mathrm{T}}(L+L^{\mathrm{T}})\tilde{\boldsymbol{p}}_e \\ &\le 0\end{aligned} \tag{44}$$

where it is noted that $A+A^{\mathrm{T}}=\mathbf{0}$, $PS+S^{\mathrm{T}}P\le\mathbf{0}$ and $L+L^{\mathrm{T}}>\mathbf{0}$. $\dot V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}})\le 0$ implies that the augmented closed-loop system (39)–(42) is uniformly stable, and $V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}})$ is nonincreasing in t and bounded. Then, $V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}})$ has a finite limit as $t\to\infty$, which implies that $\boldsymbol{p}_e$, $\tilde{\boldsymbol{p}}_e$, $\tilde{\boldsymbol{z}}$, $\tilde{\boldsymbol{\varepsilon}}$, and $\int_{t_0}^{t}\dot V(\tau,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}})\mathrm{d}\tau$ also have finite limits as $t\to\infty$. Since $V(t,\boldsymbol{p}_e(t),\theta_e(t),\tilde{\boldsymbol{p}}_e(t),\tilde{\boldsymbol{z}}(t))\le V(t_0,\boldsymbol{p}_e(t_0),\theta_e(t_0),\tilde{\boldsymbol{p}}_e(t_0),\tilde{\boldsymbol{z}}(t_0))$, then $\boldsymbol{p}_e$, $\tilde{\boldsymbol{p}}_e$, $\tilde{\boldsymbol{z}}$, and $\tilde{\boldsymbol{\varepsilon}}$ are bounded, and $\boldsymbol{p}-\boldsymbol{p}_0$, $\dot{\boldsymbol{p}}_e$, $\dot\theta_e$, $\dot{\tilde{\boldsymbol{p}}}_e$, $\dot{\tilde{\boldsymbol{z}}}$ and $\dot{\tilde{\boldsymbol{\varepsilon}}}$ are also bounded. Thus, $\ddot V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}})$ is bounded and $\dot V(t,\boldsymbol{p}_e,\theta_e,\tilde{\boldsymbol{p}}_e,\tilde{\boldsymbol{z}})$ is uniformly continuous in t. By Barbalat's Lemma,

$$\lim_{t\to\infty}x_e(t)=0,\ \lim_{t\to\infty}\sin\tfrac{\theta_e(t)}{2}=0,\ \lim_{t\to\infty}\tilde{\boldsymbol{p}}_e(t)=\mathbf{0}. \tag{45}$$

Next, we employ the extended Barbalat's Lemma (Lemma A.1) to show $\lim_{t\to\infty}y_e(t)=0$.

Define the following two functions

$$\mu_1=x_e+\tilde x_e,\ \ \mu_2=y_e\sin\theta_e. \tag{46}$$

By (39) and (41), $\dot\mu_1$ and $\dot\mu_2$ can be written in the form of $\dot\mu_1=h_{11}+h_{12}$ and $\dot\mu_2=h_{21}+h_{22}$, where

$$h_{11}=(\omega+\hat\varepsilon_2)y_e=(\omega_0+\frac{k_2v_0y_e\cos\frac{\theta_e}{2}}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}})y_e, \tag{47}$$

$$\begin{aligned}h_{12}=&(\omega+\hat\varepsilon_2)\tilde y_e-v_0(\cos\theta_e-1)+\frac{k(\tilde x_e+x_e)}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}}\\&+\frac{v_0(x_e(\cos\theta_e-1)+y_e\sin\theta_e)(2x_e+\tilde x_e)}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}(\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}+\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e})}\\&+\frac{k_1k_2x_e}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}\sqrt{1+\boldsymbol{p}_e^{\mathrm{T}}\boldsymbol{p}_e}}-[1\ \ 0]L\tilde{\boldsymbol{p}}_e,\end{aligned} \tag{48}$$

$$h_{21}=y_e\cos\theta_e(\tilde\varepsilon_2-\frac{k_2v_0y_e\cos\frac{\theta_e}{2}}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}}), \tag{49}$$

$$h_{22}=\dot y_e\sin\theta_e+y_e\cos\theta_e\frac{k_2v_0x_e\sin\frac{\theta_e}{2}+k_3\sin\frac{\theta_e}{2}}{\sqrt{1+\hat{\boldsymbol{p}}_e^{\mathrm{T}}\hat{\boldsymbol{p}}_e}}. \tag{50}$$

It follows from (45), (48), and (50) that $\lim_{t\to\infty}h_{12}(t)=0$ and $\lim_{t\to\infty}h_{22}(t)=0$. Since $\boldsymbol{p}_e$, $\tilde{\boldsymbol{p}}_e$, and $\tilde{\boldsymbol{\varepsilon}}$ are bounded and $\boldsymbol{\varepsilon}$ is also bounded under assumption [A1], it follows from assumption [A2], (39), (40), (47), and (49) that $\dot h_{11}(t)$ and $\dot h_{21}(t)$ exist and are bounded. Then, $h_{11}(t)$ and $h_{21}(t)$ are uniformly continuous in t. By Lemma A.1, we have

$$\lim_{t\to\infty}h_{11}(t)=0,\ \ \lim_{t\to\infty}h_{21}(t)=0. \tag{51}$$

If $v_0(t)$ does not converge to zero and $\lim_{t\to\infty}\omega_0(t)=0$, $\lim_{t\to\infty}h_{11}(t)=\lim_{t\to\infty}\frac{k_2v_0(t)y_e^2(t)}{\sqrt{1+\hat y_e^2(t)}}=0$ leads to $\lim_{t\to\infty}y_e(t)=0$.

If $\lim_{t\to\infty}v_0(t)=0$ and $\omega_0(t)$ does not converge to zero, $\lim_{t\to\infty}h_{11}(t)=\lim_{t\to\infty}\omega_0(t)y_e(t)=0$ leads to $\lim_{t\to\infty}y_e(t)=0$.

If neither $v_0(t)$ nor $\omega_0(t)$ converges to zero, it follows from (51) that $\lim_{t\to\infty}(h_{11}(t)+h_{21}(t))=0$, i.e.,

$$\lim_{t\to\infty}(y_e(t)(\omega_0(t)+\tilde\varepsilon_2(t)))=0. \tag{52}$$

Note that $\lim_{t\to\infty}\tilde\varepsilon_2(t)$ exists and is finite. Then, if $\lim_{t\to\infty}\omega_0(t)$ does not exist, it follows from (52) that $\lim_{t\to\infty}y_e(t)=0$. If $\omega_0(t)$ has a finite and non-zero limit and $\lim_{t\to\infty}v_0(t)$ does not exist, $\lim_{t\to\infty}h_{11}(t)=\lim_{t\to\infty}((\omega_0(t)+\frac{k_2v_0(t)y_e(t)}{\sqrt{1+\hat y_e^2(t)}})y_e(t))=0$ leads to $\lim_{t\to\infty}y_e(t)=0$. If both $v_0(t)$ and $\omega_0(t)$ have finite and non-zero limits, the selection of k_2 in this case yields $\lim_{t\to\infty}h_{11}(t)=\lim_{t\to\infty}\omega_0(t)(1+\frac{\delta y_e(t)}{\sqrt{1+\hat y_e^2(t)}})y_e(t)=0$ with $|\delta|\le 1$. Since $1+\frac{\delta y_e}{\sqrt{1+\hat y_e^2}}>0$ for $y_e\in\mathbb{R}$, then $\lim_{t\to\infty}y_e(t)=0$.

Hence, if either $v_0(t)$ or $\omega_0(t)$ does not converge to zero as $t\to\infty$, then $\lim_{t\to\infty}y_e(t)=0$.

This completes the proof. ∎

Remark 3.4: Assumption [A2] is required to facilitate the extended Barbalat's Lemma. Moreover, the P.E. condition that either $v_0(t)$ or $\omega_0(t)$ does not converge to zero, is the same as that in [8], and is needed to show $\lim_{t\to\infty}y_e(t)=0$. ∎

Remark 3.5: By setting a circular trajectory with radius r around the given target as the reference trajectory, the objective of Problem 1 can be achieved by control law (34)–(37) for Problem 2. However, it follows from (5) and (7) that the definitions of Problems 1 and 2 are essentially different in the sense that θ_e is not required for the solution to Problem 1. As θ_e is not used in control law (15)–(18), from a practical view of point, the implementation of control law (15)–(18) is easier than that of control law (34)–(37). ∎

Remark 3.6: For the trajectory tracking control problem, several kinds of bounded kinematic disturbances were considered in [14]–[17]. In this paper, we consider the input disturbances of the mobile robot as in [18]–[20]. Full rejection with respect to a wider class of input disturbances than that in [18] and [19] is achieved. ∎

Remark 3.7: In practice, some disturbances are generated by system (2) with a non-constant matrix S. For instance, disturbances caused by motor vibration depend on the velocity, that is, S is a function of v or $\dot{\boldsymbol{p}}$. In some scenarios, the model of system (2) may not be known, i.e., matrix S may be unknown. These cases are out of scope of this paper, but are our on-going or future research topics. ∎

IV. An Illustrative Example

In this section, we present an example of a mobile robot (1) with an input disturbance $\boldsymbol{\varepsilon}(t)$ in target enclosing and trajectory tracking respectively. The disturbance $\boldsymbol{\varepsilon}(t)$ is caused by an external environmental perturbation as given in the example of Remark 2.2. According to (11), the positive-definite matrix P can be chosen as $P=\begin{bmatrix}2&-1&0\\-1&5&0\\0&0&1\end{bmatrix}$. The initial state of the robot is given by $[x(0)\ y(0)\ \theta(0)]^{\mathrm{T}}=[5\ 5\ \frac{\pi}{10}]^{\mathrm{T}}$.

First, consider a target located at $\boldsymbol{p}_0=[0\ \ 0]^{\mathrm{T}}$ and a desired radius $r=3$. Set $v_0=5$, $k=10$ and $L=\mathrm{diag}(1,1)$. It is shown in Fig. 1 that under control law (15)–(18), $\boldsymbol{e}(t)$ can converge to zero, i.e., the objective of target enclosing (6) can be achieved.

Then, consider a reference trajectory generated by (3) with $\boldsymbol{p}_0(0)=[18\ 18]^{\mathrm{T}}$, $\theta_0(0)=\frac{\pi}{5}$, $v_0(t)=\frac{5}{4}-\frac{1}{4}\cos\frac{1}{4}t$, and $\omega_0(t)=\frac{1}{4}\cos\frac{1}{2}t$. Set $k_1=1$, $k_2=2$, $k_3=1$, and $L=\mathrm{diag}(100,100)$. It is shown in Fig. 2 and Fig. 3 that under control law (34)–(37), $\boldsymbol{p}_e(t)$ and $\theta_e(t)$ can converge to zero, i.e., the objective of trajectory tracking (8) can be achieved.

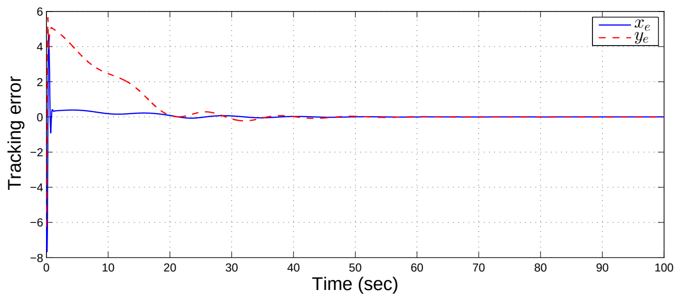

Fig. 1. Tracking error $\boldsymbol{e}$ in target enclosing during 0-100s.

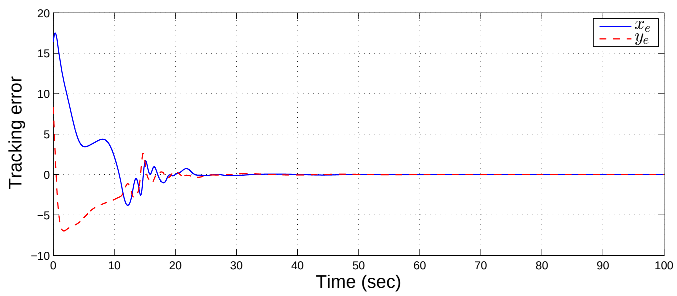

Fig. 2. Tracking error $\boldsymbol{p}_e$ in trajectory tracking during 0-100s.

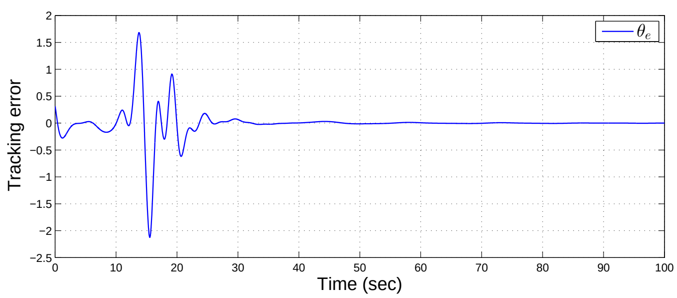

Fig. 3. Tracking error θ_e in trajectory tracking during 0-100s.

All these simulation results verify the effectiveness of the proposed control laws.

V. Conclusions

In this letter, we have proposed two dynamic control laws such that a mobile robot can enclose a given target or track a reference trajectory in the presence of a class of input disturbances. The concerned disturbance can describe linear combinations of a finite number of sinusoidal signals and step signals. Disturbance rejection can be achieved with the proposed control laws. For the future work, we will investigate the disturbance rejection in moving-target enclosing of a mobile robot and the coordination of multiple mobile robots.

Appendix

The following lemma is known as the extended Barbalat's Lemma [22, Lemma A.14].

Lemma A.1 ([22, Lemma A.14]): If a differentiable function $f(t) \in \mathbb{R}$ has a finite limit as $t \to \infty$, and its time derivative can be written as $\dot{f}(t) = g_1(t) + g_2(t)$, where $g_1(t)$ is a uniformly continuous function and $\lim_{t\to\infty} g_2(t) = 0$, then $\lim_{t\to\infty} \dot{f}(t) = 0$ and $\lim_{t\to\infty} g_1(t) = 0$. ■

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