Distributed Circular Formation Control of Ring-Networked Nonholonomic Vehicles ⋆

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Abstract

This paper investigates the circular formation control problem of multiple nonholonomic vehicles of unicycle type. The measurement of each vehicle is based on its local coordinate frame and the communication network among vehicles is modeled by a directed cycle graph. A distributed dynamic control law is designed by only using the local measurement of each vehicle and information of its neighbors in the network. The proposed control law guarantees that all vehicles move along a common circle with the given center and radius with a prespecified angular velocity, and maintain evenly spaced along the circle. Furthermore, the velocity constraint including positive-minimum linear velocity of each vehicle is explicitly taken into account.

Key words: Circular formation; Cyclic pursuit; Distributed control; Nonholonomic vehicles; Velocity constraint.

1 Introduction

Circular formation control aims at designing a distributed control law such that a group of vehicles travel along a common circle and maintain evenly spaced. In practice, circular formation can be applied in the scenario where vehicles are required to enclose, capture, secure or monitor a target, e.g., mobile sensors for ocean sampling (Leonard et al., 2007).

Many efforts have been devoted to the study of circular formation control. Sepulchre et al. (2007, 2008) presented a comprehensive investigation on the circular formation of multiple vehicles with identical unit linear velocity. In Sepulchre et al. (2007), gradient control laws based on potential functions were proposed for vehicles with all-to-all communication, which was generalized to vehicles with limited communication in Sepulchre et al. (2008). Chen & Zhang (2011) studied the collective circular motion under a jointly connected condition. An average system was used to approximate the closed-loop system by ignoring the first order of smallness $O(1/\omega_0)$, where ω_0 is the steady-state angular velocity. Then, stability analysis was given on the approximated system. Later, Chen & Zhang (2013) further considered the case where each vehicle has a local coordinate frame. El-Hawwary & Maggiore (2013) formulated the circular formation control problem as a set stabilization problem and proposed a hierarchical design approach.

Another research direction on this topic is to consider the ring-networked vehicles, which needs minimum communication links. Marshall et al. (2004, 2006a) considered the formation of multiple vehicles in cyclic pursuit and gave stability analysis on the linearized system. It was shown that the equilibrium formations of multi-vehicle systems are generalized regular polygon formations. Sinha & Ghose (2007) considered the cyclic pursuit problem of vehicles with heterogenous constant linear velocities. Later, Zheng et al. (2009) proposed a projection-based cyclic pursuit control law such that the trajectories of vehicles will never diverge. It is also noted that several works studied the case where all vehicles know the center and radius of the common circle and the common steady-state velocity. In particular, Ceccarelli et al. (2008) considered the vehicle of which the onboard sensor has limited visibility region. In Summers et al. (2009), even spacing along a given circle was achieved by vehicles with velocity constraint. Lan et al. (2010) proposed a hybrid control law using local measurements. In addition, some results on networked double-integrators (Pavone & Frazzoli, 2007; Ren, 2009; Ramirez-Riberos et al., 2010; Sharma et al.,

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2013) can be applied to nonholonomic vehicles after local feedback linearization as shown in Ren & Atkins (2007). Recently, Seyboth et al. (2014) and Zheng et al. (2015) studied circular formations where the linear velocities of vehicles are nonidentical in the steady state.

In this paper, the circular formation is prespecified with the center and radius of the common circle and the steady-state velocity of vehicles. Each vehicle has its local coordinate frame and the communication network is modeled by a directed cycle graph. A distributed dynamic control law without using any global information is proposed to globally stabilize vehicles to the prespecified circular formation. Similar to Sepulchre et al. (2008), the proposed dynamic control law requires both local measurement and communication.

The main contribution of this paper lies in the following aspects. First, the proposed control law for a prespecified circular formation does not rely on any global information, including the center and radius of the common circle, the steady-state velocity, the total number of vehicles, as well as a global inertial frame and a common reference direction. Second, global asymptotical stability of the closed-loop system instead of a linearized system (Marshall et al., 2004, 2006a) or an approximated system (Chen & Zhang, 2011, 2013) is guaranteed. Third, the proposed control law explicitly takes into account the velocity constraint including positive-minimum linear velocity, and thus can be applied to vehicles subject to stall conditions, such as fixed-wing unmanned aerial vehicles (Ren & Beard, 2004).

This paper is organized as follows. In Section 2, the problem formulation and a technical lemma are given. In Section 3, the design procedure of the proposed control law is presented. Section 4 presents the main results and Section 5 shows the simulation results of an illustrative example. Finally, the conclusion is drawn in Section 6.

Notations: Throughout the paper, the norm $\|x\|$ of a vector $x \in \mathbb{R}^n$ is defined as $\|x\| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$.

2 Preliminaries

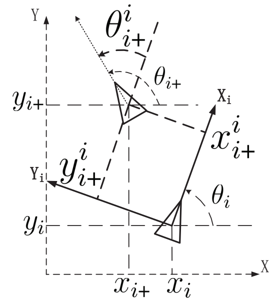
2.1 Problem Formulation

Consider N nonholonomic vehicles of unicycle type. The kinematic model of each vehicle is described by:

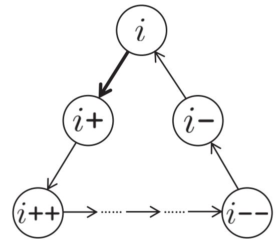
$$\dot{x}_i = v_i \cos\theta_i,\ \dot{y}_i = v_i \sin\theta_i,\ \dot{\theta}_i = \omega_i,\ i = 1, ..., N, \quad (1)$$

where $p_i := [x_i\ y_i]^{\mathrm{T}} \in \mathbb{R}^2$ is the coordinate of the center of mass (position) and $\theta_i \in \mathbb{R}$ is the heading angle (orientation) of each vehicle in the inertial frame (see Fig. 1(a)). The linear velocity $v_i \in \mathbb{R}$ and the angular velocity $\omega_i \in \mathbb{R}$ are control inputs.

Based on the pursuit graph (Marshall et al., 2006a, Definition 2), the topology of the communication network among vehicles is modeled by a directed cycle graph $\mathcal{G} = \{\mathcal{O}, \mathcal{E}\}$ which consists of a finite set of nodes $\mathcal{O} =$



(a) Local measurement



(b) Directed cycle graph $\mathcal{G}$

Fig. 1. Illustration of measurement in the coordinate frame of vehicle i when there is a directed edge $(i, i{+})$ in $\mathcal{G}$.

$\{1, ..., N\}$ representing N vehicles, and a set of edges $\mathcal{E} = \{(j, i) : i \neq j,\ i, j \in \mathcal{O}\}$ containing directed edges from node j to node i. For each node i, there exists one incoming edge $(i{-}, i)$ and one outgoing edge $(i, i{+})$, where node $i{-}$ is called the *pre-neighbor* of node i and node $i{+}$ is called the *next-neighbor* of node i (see Fig. 1(b)).

A directed edge $(i, i{+})$ means that vehicle i can send its information to vehicle $i{+}$ and can measure the states of vehicle $i{+}$ in its local coordinate frame. The coordinate frame of vehicle i has the origin at its position p_i and the x-axis coincident with its orientation θ_i. The states $p_{i+} = [x_{i+}\ y_{i+}]^{\mathrm{T}}$ and θ_{i+} measured in the coordinate frame of vehicle i is denoted by p^i_{i+} and θ^i_{i+} respectively (see Fig. 1(a)). It is noted that the information flow of this network is not strictly one-way since the relative states between vehicle i and $i{+}$ are measured by vehicle i. Each vehicle i can measure neither its state $[p_i^{\mathrm{T}}\ \theta_i]^{\mathrm{T}}$ nor the relative position $p_i - p_j$ due to a lack of a global inertial frame and a common reference direction respectively.

A circular formation requires vehicles to move with a constant angular velocity ω_c along a common circle with the center $q_c := [x_c\ y_c]^{\mathrm{T}}$ and radius r_c, and to maintain evenly spaced along the circle. The counterclockwise ($\omega_c > 0$) *circular formation* is formally defined as:

Definition 2.1 A set $\bar{\mathcal{C}}_\rho(t) = \{[\bar{p}_i^{\mathrm{T}}(t)\ \bar{\theta}_i(t)\ \bar{\omega}_i(t)\ \bar{v}_i(t)]^{\mathrm{T}} \in \mathbb{R}^5,\ i = 1, ..., N\}$ is a *circular formation* with the *formation parameter* $\rho = [q_c^{\mathrm{T}}\ r_c\ \omega_c]^{\mathrm{T}} \in \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$, if

$$\bar{p}_i(t) - q_c = r_c[\sin\bar{\theta}_i(t)\ \ -\cos\bar{\theta}_i(t)]^{\mathrm{T}}, \quad (2)$$
$$\|\bar{p}_{i(i+)}(t)\| = \|\bar{p}_{i+(i++)}(t)\| = ... = \|\bar{p}_{(i-)i}(t)\|, \quad (3)$$
$$\bar{\omega}_i(t) = \omega_c,\ \bar{v}_i(t) = \omega_c r_c, \quad (4)$$

for all $t \geq 0$, where $\bar{p}_{i(i+)} = \bar{p}_i - \bar{p}_{i+}$. ■

Now, the *circular formation control problem* considered in this paper is formally defined as follows.

Definition 2.2 Consider N vehicles (1) and the communication digraph $\mathcal{G}$, and define $\mathcal{C}(t) = \{[p_i^{\mathrm{T}}(t)\ \theta_i(t)\ \omega_i(t)\ v_i(t)]^{\mathrm{T}} \in \mathbb{R}^5,\ i = 1, ..., N\}$. Given a formation parameter $\rho = [q_c^{\mathrm{T}}\ r_c\ \omega_c]^{\mathrm{T}} \in \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$, for vehicle i, $i = 1, ..., N$, with any initial states $[p_i^{\mathrm{T}}(t_0)\ \theta_i(t_0)]^{\mathrm{T}} \in$

$\mathbb{R}^3$, $\forall t_0 \geq 0$, find a dynamic control law in the form of

$$[v_i \ \omega_i]^{\mathrm{T}} = \sigma(\hat{\rho}_i^i,\ m_i,\ \varrho_{i\text{-}}^{i\text{-}}),\ \dot{\hat{\rho}}_i^i = \kappa(\hat{\rho}_i^i,\ \varrho_{i\text{-}}^{i\text{-}}), \tag{5}$$

such that $\mathcal{C}(t)$ converges to a *circular formation* $\bar{\mathcal{C}}_\rho(t)$ as $t \to \infty$, where $\hat{\rho}_i^i$, to be designed later, is the estimate of ρ in the local coordinate frame, $\varrho_{i\text{-}}^{i\text{-}}$ is the information of its *pre-neighbor*, m_i is the local measurement, and functions $\sigma(\cdot)$ and $\kappa(\cdot)$ are both sufficiently smooth. ■

In this paper, we consider the *circular formation control problem* under the following assumption:

[A1] Only one vehicle l knows the formation parameter in its local coordinate frame at the initial time t_0, i.e., $[q_c^{l\mathrm{T}}(t_0)\ r_c\ \omega_c]^{\mathrm{T}}$.

In general, vehicle l is anonymous and other vehicles cannot identify it.

Remark 2.1 To make vehicles maintain a cyclic pursuit manner, each vehicle is only required to connect to its neighbors by onboard point-to-point communication device and sensor. In Marshall et al. (2006b), the practicality of cyclic pursuit as a distributed control strategy for multiple mobile robots was demonstrated by experiment. Since devices and sensors are usually distance-constrained in practice, vehicles need to locate within a bounded space such that the network is connected. ■

Remark 2.2 Collision avoidance is not included in this paper, and we may assume that vehicles move on different altitudes as in Seyboth et al. (2014). Approaches to collision avoidance will be explored in future. ■

2.2 A Technical Lemma

In this section, we present a technical lemma which will be used in stability analysis of the resulting closed-loop system. Consider the following system:

$$\dot{\chi} = f(\chi, d(t)) + g(\chi, \xi, d(t)), \tag{6}$$

where $\chi \in \mathbb{R}^n$ is the state, $\xi \in \mathbb{R}^m$ is an exogenous signal, $d : \mathbb{R}_{\geq 0} \mapsto \mathcal{D}$ is a time-varying function and $\mathcal{D}$ is a compact subset of $\mathbb{R}^q$. $f(\chi, d(t))$ and $g(\chi, \xi, d(t))$ are continuous in their arguments. $f(\chi, d(t))$ is locally Lipschitz on χ uniformly on d (Lin et al., 1996) and $g(\chi, \xi, d(t))$ is locally Lipschitz on (χ, ξ) uniformly on d. System (6) can be viewed as a perturbation of the nominal system

$$\dot{\chi} = f(\chi, d(t)). \tag{7}$$

Lemma 2.1 Let $\chi = 0$ be an equilibrium point for system (6). If conditions [C1]-[C3] are satisfied, system (6) is globally uniformly asymptotically stable at $\chi = 0$.

[C1] The nominal system (7) is globally uniformly asymptotically stable with a Lyapunov function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ such that for all $t \geq 0$ and all $\chi \in \mathbb{R}^n$,

$$\underline{W}(\chi) \leq V(t, \chi) \leq \overline{W}(\chi), \tag{8}$$

$$\frac{\partial V(t,\chi)}{\partial t} + \frac{\partial V(t,\chi)}{\partial \chi} f(\chi, d(t)) \leq -W(\chi), \tag{9}$$

$$\left\| \frac{\partial V(t,\chi)}{\partial \chi} \right\| \|\chi\| \leq c_1 V(t, \chi),\ \forall \|\chi\| \geq \zeta, \tag{10}$$

$$\left\| \frac{\partial V(t,\chi)}{\partial \chi} \right\| \leq c_2,\ \forall \|\chi\| \leq \zeta, \tag{11}$$

where $\underline{W}(\chi)$ and $\overline{W}(\chi)$ are two class $\mathcal{K}_\infty$ functions, $W(\chi)$ is a *positive semi-definite* function, and $c_1 > 0$, $\zeta > 0$, and $c_2 > 0$ are some constants.

[C2] There exists a class $\mathcal{KL}$ function $\beta(\cdot)$ and a class $\mathcal{K}$ function $\alpha(\cdot)$, such that for all $\xi(t_0) \in \mathbb{R}^m$, $\|\xi(t)\| \leq \beta(\|\xi(t_0)\|,\ t - t_0)$ and $\int_{t_0}^{\infty} \|\xi(t)\| dt \leq \alpha(\|\xi(t_0)\|), \forall t \geq t_0$.

[C3] The function $g(\chi, \xi, d(t))$ satisfies that for all $\chi \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^m$,

$$\|g(\chi, \xi, d(t))\| \leq \|\xi\| \left(\Theta_1(\|\xi\|) + \|\chi\| \Theta_2(\|\xi\|)\right), \tag{12}$$

where $\Theta_1,\ \Theta_2 : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ are continuous functions. ■

The proof can be found in Yu & Liu (2016) and thus omitted here. When $d(t) \equiv \mathbf{d}$, systems (6) and (7) reduce to autonomous systems. The notion of "uniformly" is not necessary and $V(t, \chi)$ can be replaced by $V(\chi)$.

3 Control Law Design

In this section, we present the control law design for the *circular formation control problem*.

To begin with, define

$$R(z) = \begin{bmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{bmatrix},\ S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \begin{array}{l} P = [0\ \ {-1}]^{\mathrm{T}}, \\ Q = [1\ \ \ 0]^{\mathrm{T}}, \end{array}$$

where it is noted that $S + S^{\mathrm{T}} = 0$.

Next, p_{i+}, θ_{i+}, and q_c measured in the coordinate frame of vehicle i can be expressed as $p_{i+}^i = R(\theta_i)(p_{i+} - p_i)$, $\theta_{i+}^i = \theta_{i_+} - \theta_i$, and $q_c^i = R(\theta_i)(q_c - p_i)$ respectively. Then,

$$\dot{q}_c^i = \omega_i S q_c^i - v_i Q. \tag{13}$$

Define the formation tracking error $p_{ei} = [x_{ei}\ y_{ei}]^{\mathrm{T}}$ as

$$p_{ei} = R(\theta_i)(q_c - p_i + r_c[\sin\theta_i\ \ -\cos\theta_i]^{\mathrm{T}}) = q_c^i + r_c P.$$

As a result, the error dynamics are

$$\dot{p}_{ei} = \omega_i S p_{ei} - (v_i - \omega_i r_c) Q. \tag{14}$$

It is noted that the objective described in (2) can be achieved if $p_e = \mathrm{col}(p_{e1}, ..., p_{eN})$ converges to zero.

When vehicles travel along a common circle, the objective described in (3) can be achieved if $\theta = \mathrm{col}(\theta_1, .., \theta_N)$ converges to $\mathbb{E} = \{\theta \in \mathbb{R}^N : (\theta_{i+} - \theta_i) \bmod 2\pi = 2\pi/N\}$.

Furthermore, $[v_i\ \omega_i]^{\mathrm{T}}$ needs to be designed such that its steady state can satisfy (4). However, note that only vehicle l has access to the formation parameter.

In what follows, the design procedure of the proposed dynamic control law is presented in three steps. First, a distributed observer is developed for vehicle i to estimate the formation parameter. Then, an angular velocity controller is designed such that all vehicles can be evenly spaced along the circle. Finally, a linear velocity controller is proposed for global convergence to the common circle with the given center and radius. Meanwhile, all velocities converge to the given common values.

3.1 Distributed Observer Design

Define an estimate of the formation parameter in the coordinate frame of vehicle i as $[\hat{q}_i^{i\mathrm{T}}\ \hat{r}_i\ \hat{\omega}_i]^{\mathrm{T}} \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. With the local measurement θ^i_{i+}, define angle ϕ_i as

$$\phi_i = \begin{cases} \theta^l_{l+} + 2\pi \ = \ \theta_{l+} - \theta_l + 2\pi, & i = l \\ \quad\ \ \theta^i_{i+} \ = \ \theta_{i+} - \theta_i, & i \neq l \end{cases}. \tag{15}$$

Then, a dynamic observer for vehicle i is designed as

$$\begin{aligned} \dot{\hat{q}}_i^i &= \omega_i S \hat{q}_i^i - v_i Q + a_i(R(\phi_{i-})\hat{q}_{i-}^{i-} - R(\phi_{i-})p_i^{i-} - \hat{q}_i^i) \\ &\quad + b_i(q_c^i - \hat{q}_i^i), & (16) \\ \dot{\hat{r}}_i &= a_i(\hat{r}_{i-} - \hat{r}_i) + b_i(r_c - \hat{r}_i), & (17) \\ \dot{\hat{\omega}}_i &= a_i(\hat{\omega}_{i-} - \hat{\omega}_i) + b_i(\omega_c - \hat{\omega}_i),\ i = 1, ..., N, & (18) \end{aligned}$$

where a_i and b_i are defined as: $a_i = 0$ and $b_i = 1$ if $i = l$; $a_i = 1$ and $b_i = 0$, otherwise. Since vehicle l has access to the formation parameter, it does not need to use the estimate from its *pre-neighbor*.

The initial state $[\hat{q}_i^{i\mathrm{T}}(t_0)\ \hat{r}_i(t_0)\ \hat{\omega}_i(t_0)]^{\mathrm{T}}$ can be arbitrarily selected in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. The observer design is only based on the local information $[\hat{q}_i^{i\mathrm{T}}\ \hat{r}_i\ \hat{\omega}_i]^{\mathrm{T}}$ and information from its *pre-neighbor* vehicle $i-$, i.e., $[\hat{q}_{i-}^{i-\mathrm{T}}\ \hat{r}_{i-}\ \hat{\omega}_{i-}]^{\mathrm{T}}$ and $[p_i^{i-\mathrm{T}}\ \phi_{i-}]^{\mathrm{T}}$. Thus, observer (16)-(18) is a distributed one and the inter-vehicle communication can be implemented as follows. First, the sensor of vehicle i measures p^i_{i+} and θ^i_{i+} in the local coordinate frame. Then, the communication device of vehicle i sends the local information $[\hat{q}_i^{i\mathrm{T}}\ \hat{r}_i\ \hat{\omega}_i]^{\mathrm{T}}$ and the local measurement $[p_{i+}^{i\mathrm{T}}\ \phi_i]^{\mathrm{T}}$ to vehicle $i+$. Moreover, the distributed observer (16)-(18) has a property as is stated in the following lemma.

Lemma 3.1 Consider N *distributed dynamic observers* (16)-(18) and the communication digraph $\mathcal{G}$. Given a formation parameter $[q_c^{\mathrm{T}}\ r_c\ \omega_c]^{\mathrm{T}} \in \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$, under assumption [A1], for any initial states $[\hat{q}_i^{i\mathrm{T}}(t_0)\ \hat{r}_i(t_0)\ \hat{\omega}_i(t_0)]^{\mathrm{T}} \in \mathbb{R}^4$, $\forall t_0 \geq 0$, $i = 1, ..., N$, $[\hat{q}_i^{i\mathrm{T}}(t)\ \hat{r}_i(t)\ \hat{\omega}_i(t)]^{\mathrm{T}}$ converge to $[q_c^{i\mathrm{T}}(t)\ r_c\ \omega_c]^{\mathrm{T}}$ *exponentially* as $t \to \infty$. ∎

The proof is given in Appendix A.

3.2 Controller Design for Angular Velocity

In order to achieve objectives described in (3) and (4), the angular velocity controller for vehicle i is designed as

$$\omega_i = \hat{\omega}_i + k_\omega \tanh(\mu(\phi_i - \phi_{i-})), \tag{19}$$

where ϕ_i is defined in (15), $k_\omega > 0$ and $\mu > 0$ are design parameters, and function $\tanh(\cdot)$ is used for achieving bounded control input. For vehicle i, ϕ_i and ϕ_{i-} are obtained from the local measurement and the communication from its *pre-neighbor* respectively. Then, we have the following lemma.

Lemma 3.2 Consider the communication digraph $\mathcal{G}$, and systems $\dot{\theta}_i = \omega_i,\ i = 1, ..., N$, with the control law (18)-(19). Under assumption [A1], for any initial states $\theta_i(t_0) \in \mathbb{R}$, $i = 1, ..., N$, $\forall t_0 \geq 0$, $(\theta_{i+}(t) - \theta_i(t)) \bmod 2\pi$ converge to $2\pi/N$ asymptotically as $t \to \infty$. Meanwhile, $\omega_i(t)$, $i = 1, ..., N$, converge to ω_c. ∎

Proof: Define

$$\varphi_i = \phi_{i+} - \phi_i. \tag{20}$$

Using (15), (19), and (20) yields

$$\begin{aligned} \dot{\varphi}_i &= k_\omega(\tanh(\mu\varphi_{i+}) - 2\tanh(\mu\varphi_i) + \tanh(\mu\varphi_{i-})) \\ &\quad + \hat{\omega}_{i+} - 2\hat{\omega}_i + \hat{\omega}_{i-}. \end{aligned} \tag{21}$$

Denote $\varphi = \mathrm{col}(\varphi_l, \varphi_{l+}, ..., \varphi_{l-})$ and $\mathrm{T}(\varphi) = \mathrm{col}(\tanh(\mu\varphi_l), \tanh(\mu\varphi_{l+}), ..., \tanh(\mu\varphi_{l-}))$. The system consisting of N systems in the form of (21) can be written in the following compact form:

$$\dot{\varphi} = -k_\omega L\, \mathrm{T}(\varphi) - L\tilde{\omega}, \tag{22}$$

where L is the Laplacian matrix of the underlying graph of $\mathcal{G}$ and $\tilde{\omega} = \mathrm{col}(\tilde{\omega}_l, \tilde{\omega}_{l+}, ..., \tilde{\omega}_{l-})$ with $\tilde{\omega}_i = \hat{\omega}_i - \omega_c$.

It is noted that (22) is in the form of (6). Lemma 2.1 will be used to prove that $\varphi(t) \to 0$ as $t \to \infty$.

Firstly, we will show that the nominal system $\dot{\varphi} = -k_\omega L\, \mathrm{T}(\varphi)$ is globally asymptotically stable. Motivated by Ren (2008), consider the following Lyapunov function candidate $V(\varphi) = \sum_{i=1}^N \log(\cosh(\mu\varphi_i))$, which is positive definite and radially unbounded with respect to φ. Taking the time derivative of $V(\varphi)$ along the trajectories of system $\dot{\varphi} = -k_\omega L\, \mathrm{T}(\varphi)$ gives

$$\begin{aligned} \dot{V}(\varphi) &= \mu \textstyle\sum_{i=1}^N \dot{\varphi}_i \tanh(\mu\varphi_i) \\ &= -\mu k_\omega \textstyle\sum_{i=1}^N (-\tanh(\mu\varphi_i)\tanh(\mu\varphi_{i-}) \\ &\quad - \tanh(\mu\varphi_i)\tanh(\mu\varphi_{i+}) + 2\tanh^2(\mu\varphi_i)) \\ &= -\mu k_\omega \textstyle\sum_{i=1}^N (\tanh(\mu\varphi_{i+}) - \tanh(\mu\varphi_i))^2 \leq 0. \end{aligned}$$

Let $\mathcal{S} = \{\varphi | \dot{V}(\varphi) \equiv 0\}$. Note that $\dot{V}(\varphi) \equiv 0$ implies $\tanh(\mu\varphi_1) \equiv \tanh(\mu\varphi_2) \equiv ... \equiv \tanh(\mu\varphi_N)$. Since $\tanh(\cdot)$ is strictly increasing in $(-\infty, +\infty)$, then $\mathcal{S} = \{\varphi | \varphi_1 \equiv \varphi_2 \equiv ... \equiv \varphi_N\}$. It follows from the definition of φ_i in (20) that $\varphi_1 + \varphi_2 + ... + \varphi_N \equiv 0$. Thus, $\mathcal{S} = \{\varphi | \varphi \equiv 0\}$ and $\dot{\varphi} \equiv 0$ for $\varphi \in \mathcal{S}$. By LaSalle's Invariance Principle, $\varphi(t) \to 0$ asymptotically as $t \to \infty$. Hence, system $\dot{\varphi} = -k_\omega L\, \mathrm{T}(\varphi)$ is globally asymptotically stable at $\varphi = 0$. Noting that $V(\varphi)$ satisfies (8)-(11), condition [C1] in Lemma 2.1 is satisfied.

Secondly, Lemma 3.1 holds by assumption [A1] and then $\tilde{\omega} \to 0$ exponentially as $t \to \infty$, which implies that condition [C2] in Lemma 2.1 is satisfied.

Thirdly, $-L\tilde{\omega}$ satisfies condition [C3] in Lemma 2.1.

Hence, by Lemma 2.1, system (22) is globally asymptotically stable at $\varphi = 0$, and it follows from (20) that

$$\lim_{t\to\infty} \phi_1(t) = \lim_{t\to\infty} \phi_2(t) = ... = \lim_{t\to\infty} \phi_N(t). \tag{23}$$

From the definition of ϕ_i in (15), we have $\sum_{i=1}^{N} \phi_i \equiv 2\pi$ and thus $\lim_{t\to\infty} \phi_i(t) = 2\pi/N$. Then, $\lim_{t\to\infty} (\theta_{l+}(t) - \theta_l(t)) = 2\pi/N - 2\pi$ and $\lim_{t\to\infty} (\theta_{i+}(t) - \theta_i(t)) = 2\pi/N,\ i \neq l$, i.e.,

$$\lim_{t\to\infty} ((\theta_{i+}(t) - \theta_i(t)) \bmod 2\pi) = 2\pi/N,\ i = 1, .., N. \tag{24}$$

Furthermore, using Lemma 3.1, (19) and (23), one can conclude that $\omega_i(t)$ converges to ω_c as $t \to \infty$. The proof is thus completed. ■

Remark 3.1 It can be observed from (19) that the controller design is independent of the total number of vehicles N. Thus, if the ring structure of the network is maintained, Lemma 3.2 holds when some vehicles are added into the group or removed from the group. ■

3.3 Controller Design for Linear Velocity

In order to achieve objectives described in (2) and (4), the linear velocity controller for vehicle i is designed as

$$v_i = \omega_i \hat{r}_i + k_v \tanh(\nu Q^{\mathrm{T}}(\hat{q}_i^i + \hat{r}_i P)), \tag{25}$$

where $k_v > 0$ and $\nu > 0$ are design parameters, and function $\tanh(\cdot)$ is used for achieving bounded control input. Then, we have the following lemma.

Lemma 3.3 Consider the communication digraph $\mathcal{G}$, and N systems in the form of (14) with the control law consisting of (16)-(18), (19), and (25). Under assumption [A1], for any initial states $p_{ei}(t_0) \in \mathbb{R}^2$, $i = 1, ..., N$, $\forall t_0 \geq 0$, $p_{ei}(t)$ converge to 0 asymptotically as $t \to \infty$. Meanwhile, $v_i(t)$, $i = 1, ..., N$, converge to $\omega_c r_c$. ■

Proof: Define $\hat{p}_{ei} = [\hat{x}_{ei}\ \hat{y}_{ei}]^{\mathrm{T}}$ as $\hat{p}_{ei} = \hat{q}_i^i + \hat{r}_i P$. Then, (25) can be rewritten as $v_i = w_i \hat{r}_i + k_v \tanh(\nu \hat{x}_{ei})$.

Defining $\tilde{p}_{ei} = [\tilde{x}_{ei}\ \tilde{y}_{ei}]^{\mathrm{T}}$ as $\tilde{p}_{ei} = \hat{p}_{ei} - p_{ei}$ gives

$$\tilde{p}_{ei} = \hat{q}_i^i - q_c^i + (\hat{r}_i - r_c)P = \tilde{q}_i + \tilde{r}_i P. \tag{26}$$

Let $\xi_i = [\tilde{p}_{ei}^{\mathrm{T}}\ \tilde{r}_i]^{\mathrm{T}} = [\tilde{x}_{ei}\ \tilde{y}_{ei}\ \tilde{r}_i]^{\mathrm{T}}$. The closed-loop system consisting of system (14) and the control law composed of (16)-(18), (19), and (25) can be written as

$$\dot{p}_{ei} = f(p_{ei}, \omega_i(t)) + g(p_{ei}, \xi_i, \omega_i(t)), \tag{27}$$

where

$$f(p_{ei}, \omega_i(t)) = \omega_i S p_{ei} - (v_{ei} - \omega_i r_c)Q, \tag{28}$$

$$g(p_{ei}, \xi_i, \omega_i(t)) = (v_{ei} - v_i)Q, \tag{29}$$

with $v_{ei} = w_i r_c + k_v \tanh(\nu x_{ei})$. Moreover, $v_{ei} - v_i$ can be expressed as

$$v_{ei} - v_i = k_v(\tanh(\nu x_{ei}) - \tanh(\nu(x_{ei} + \tilde{x}_{ei}))) - \omega_i \tilde{r}_i. \tag{30}$$

By (19) and Lemma 3.1, $\omega_i(t)$ is uniformly bounded and thus $\omega_i(t) \in \Omega_i$ with a compact set Ω_i. It is noted that (27) is in the form of (6). In what follows, Lemma 2.1 will be used to prove that $p_{ei}(t) \to 0$ as $t \to \infty$.

Firstly, we will show that the nominal system $\dot{p}_{ei} = f(p_{ei}, \omega_i(t))$ is globally uniformly asymptotically stable. Consider a Lyapunov function candidate $V_i(t, p_{ei}) : \mathbb{R}_{\geq 0} \times \mathbb{R}^2 \mapsto \mathbb{R}_{\geq 0}$ as $V_i(t, p_{ei}) = \frac{1}{2} p_{ei}^{\mathrm{T}} p_{ei}$, which is positive-definite, decrescent and radially unbounded. Taking the time derivative of $V_i(t, p_{ei})$ along the trajectories of the system $\dot{p}_{ei} = f(p_{ei}, \omega_i(t))$ yields

$$\begin{aligned} \dot{V}_i(t, p_{ei}) &= \tfrac{1}{2}\omega_i p_{ei}^{\mathrm{T}}(S + S^{\mathrm{T}})p_{ei} - k_v x_{ei} \tanh(\nu x_{ei}) \\ &= -k_v x_{ei} \tanh(\nu x_{ei}) \leq 0. \end{aligned}$$

Thus, $V_i(t, p_{ei})$ is nonincreasing in t and bounded, which implies that $\lim_{t\to\infty} \int_0^t \dot{V}_i(\tau, p_{ei}) d\tau$ exists and is finite. Since $V_i(t, p_{ei}) \leq V_i(t_0, p_{ei}(t_0))$, $\forall t \geq t_0$, p_{ei} is bounded and $\dot{p}_{ei}$ is also bounded. Then, $\ddot{V}_i(t, p_{ei})$ is bounded in t and $\dot{V}_i(t, p_{ei})$ is uniformly continuous in t. By Barbalat's Lemma, $\lim_{t\to\infty} x_{ei}(t) = 0$.

Using (28) leads to $\dot{x}_{ei} = \omega_i y_{ei} - k_v \tanh(\nu x_{ei})$. Let $h_1(t) = \omega_i y_{ei}$ and $h_2(t) = -k_v \tanh(\nu x_{ei})$. Since $\lim_{t\to\infty} x_{ei}(t) = 0$, we have $\lim_{t\to\infty} h_2(t) = 0$. It follows from (19)-(20) that $\dot{\omega}_i = \dot{\hat{\omega}}_i + \mu k_\omega \dot{\varphi}_{i\text{-}} \operatorname{sech}^2(\mu\varphi_{i\text{-}})$. Using Lemma 3.1 and (21), $\dot{\hat{\omega}}_i$ and $\dot{\varphi}_{i\text{-}}$ are bounded. Then, $\dot{\omega}_i$ is bounded. Since y_{ei} and $\dot{y}_{ei}$ are bounded, $\dot{h}_1(t)$ is bounded and $h_1(t)$ is uniformly continuous in t. By the extended Barbalat's Lemma (Dixon et al., 2001, Lemma A.14), $\lim_{t\to\infty} h_1(t) = 0$. Noting that $\lim_{t\to\infty} \omega_i(t) = \omega_c > 0$ from Lemma 3.2, one can conclude that $\lim_{t\to\infty} y_{ei}(t) = 0$.

Hence, the nominal system $\dot{p}_{ei} = f(p_{ei}, \omega_i(t))$ is globally uniformly asymptotically stable at $p_{ei} = 0$. Noting that the Lyapunov function $V_i(t, p_{ei})$ satisfies (8)-(11), condition [C1] in Lemma 2.1 is satisfied.

Secondly, it follows from Lemma 3.1 and (26) that $\xi_i(t)$ converges to 0 exponentially as $t \to \infty$. Thus, condition [C2] in Lemma 2.1 is satisfied.

Thirdly, it follows from (29) and (30) that

$$\begin{aligned} \|g(p_{ei}, \xi_i, \omega_i(t))\| &= |v_{ei} - v_i| \leq 2k_v\nu|\tilde{x}_{ei}| + |\tilde{r}_i||\omega_i| \\ &\leq |\xi_i|(2k_v\nu + \sup\Omega_i), \end{aligned} \tag{31}$$

which satisfies condition [C3] in Lemma 2.1.

Therefore, by Lemma 2.1, system (27) is globally uniformly asymptotically stable at $p_{ei} = 0$. Then, using (25) and Lemmas 3.1 and 3.2, one can conclude that $v_i(t)$ converges to $\omega_c r_c$ as $t \to \infty$. The proof is thus completed. ■

4 Main Results

The proposed control law consisting of (16)-(18), (19), and (25) is only based on the local information $[\hat{q}_i^{i\mathrm{T}}\ \hat{r}_i\ \hat{\omega}_i]^{\mathrm{T}}$, local measurement ϕ_i and information from its *pre-neighbor* vehicle i-, i.e., $[\hat{q}_{i\text{-}}^{i\text{-}\mathrm{T}}\ \hat{r}_{i\text{-}}\ \hat{\omega}_{i\text{-}}]^{\mathrm{T}}$ and $[p_i^{i\text{-}\mathrm{T}}\ \phi_{i\text{-}}]^{\mathrm{T}}$, and thus can be described in the form of (5). Now, we are ready to present the main result as follows.

Theorem 4.1 The distributed dynamic control law consisting of (16)-(18), (19), and (25) solves the *circular formation control problem* under assumption [A1]. ■

Proof: To prove Theorem 4.1, we need to establish stability analysis on two augmented closed-loop systems sequentially. First, consider the closed-loop system Σ^1 consisting of N systems $\dot{\theta}_i = \omega_i$ and the control law (18)-(19). Using Lemma 2.1, the fact that $\theta = \mathrm{col}(\theta_1, .., \theta_N)$ converges to the set $\mathbb{E}$ is proved in Lemma 3.2. Then, consider the closed-loop system Σ^2 consisting of N systems in the form of (14) and the control law composed of (16)-(18), (19), and (25). Using Lemmas 2.1 and 3.2, the result that $p_e = \mathrm{col}(p_{e1}, ..., p_{eN})$ converges to zero is established in Lemma 3.3. In addition, it follows from Lemmas 3.2 and 3.3 that the objective described in (4) can also be achieved. Thus, the proof of Theorem 4.1 follows directly from those of Lemmas 3.2 and 3.3. ■

Remark 4.1 To show stability of these two augmented closed-loop systems Σ^1 and Σ^2, one typical way is to find two appropriate Lyapunov function candidates $V_1(\theta, \tilde{\omega})$ and $V_2(p_e, \tilde{q}, \tilde{r}, \tilde{\omega})$ for Σ^1 and Σ^2 respectively, where $\tilde{q} = \mathrm{col}(\tilde{q}_1 ..., \tilde{q}_N)$, $\tilde{r} = \mathrm{col}(\tilde{r}_1, ..., \tilde{r}_N)$, and $\tilde{\omega} = \mathrm{col}(\tilde{\omega}_1, ..., \tilde{\omega}_N)$ with $\tilde{q}_i = \hat{q}_i^i - q_c^i$, $\tilde{r}_i = \hat{r}_i - r_c$, and $\tilde{\omega}_i = \hat{\omega}_i - \omega_c$. However, it is not easy to find such Lyapunov function candidates, especially $V_2(p_e, \tilde{q}, \tilde{r}, \tilde{\omega})$ for Σ^2. To overcome this difficulty, Lemma 2.1 is developed. Then, we rewrite Σ^1 and Σ^2 in the form of (6), and use Lemma 2.1 to prove Lemmas 3.2 and 3.3 from which Theorem 4.1 follows. In this way, there is no need to find $V_1(\theta, \tilde{\omega})$ and $V_2(p_e, \tilde{q}, \tilde{r}, \tilde{\omega})$. The key of using Lemma 2.1 is to consider the closed-loop system as a perturbed system and the perturbation results from the inter-vehicle communication. ■

More interestingly, consider the case where each vehicle is subject to the following velocity constraint:

$$v_i \in [v_{\min},\ v_{\max}],\ \omega_i \in [-\omega_{\max},\ \omega_{\max}], \tag{32}$$

where $v_{\max} > v_{\min} > 0$ and $\omega_{\max} > 0$ are known constants. Constraint (32) can be used to describe the velocity constraint of vehicles subject to stall conditions, e.g., fixed-wing unmanned aerial vehicles (Ren & Beard, 2004). In this case, the following assumption is needed.

[A2] $r_c \in [\underline{r}_c,\ \overline{r}_c]$ and $\omega_c \in [\underline{\omega}_c,\ \overline{\omega}_c]$, where $\overline{r}_c > \underline{r}_c > 0$ and $\overline{\omega}_c > \underline{\omega}_c > 0$ are known constants satisfying $\overline{\omega}_c < \omega_{\max}$, $\overline{\omega}_c \overline{r}_c < v_{\max}$ and $\underline{\omega}_c \underline{r}_c > v_{\min}$.

Assumption [A2] requires that r_c and ω_c are properly bounded with respect to the bounds of the velocity constraint (32). Then, we have the following proposition.

Proposition 4.1 Under assumption [A2], the velocity constraint (32) can be always satisfied if $[\hat{r}_i(t_0)\ \hat{\omega}_i(t_0)]^{\mathrm{T}} \in [\underline{r}_c,\ \overline{r}_c] \times [\underline{\omega}_c,\ \overline{\omega}_c]$, $i = 1, ..., N$, $\forall t \geq t_0$, and the parameters μ, ν, k_ω and k_v are chosen as $\mu, \nu > 0$ and

$$0 < k_\omega < \min(\Delta \underline{v}/\underline{r}_c,\ \Delta \overline{v}/\overline{r}_c,\ \omega_{\max} - \overline{\omega}_c), \tag{33}$$
$$0 < k_v \leq \min(\Delta \underline{v} - k_\omega \underline{r}_c,\ \Delta \overline{v} - k_\omega \overline{r}_c), \tag{34}$$

where $\Delta \underline{v} = \underline{\omega}_c \underline{r}_c - v_{\min}$ and $\Delta \overline{v} = v_{\max} - \overline{\omega}_c \overline{r}_c$. ■

Proof: Letting $l = 1$ and using induction, we obtain

$$\tilde{r}_i(t) = e^{-(t-t_0)} \sum_{k=1}^{i} \frac{(t-t_0)^{k-1}}{(k-1)!} \tilde{r}_{i-k+1}(t_0), \forall i \in [1, N], \tag{35}$$

from (A.1). Note that $e^{-(t-t_0)} \sum_{k=1}^{i} \frac{(t-t_0)^{k-1}}{(k-1)!} \in (0, 1]$. For all $\tilde{r}_i(t_0) \leq \overline{r}_c - r_c$, $i = 1, ..., N$, (35) and $\overline{r}_c - r_c \geq 0$ lead to $\tilde{r}_i(t) \leq \overline{r}_c - r_c$, $\forall t \geq t_0$, while for all $\tilde{r}_i(t_0) \geq \underline{r}_c - r_c$, (35) and $\underline{r}_c - r_c \leq 0$ lead to $\tilde{r}_i(t) \geq \underline{r}_c - r_c$, $\forall t \geq t_0$. Thus, $\hat{r}_i(t) \in [\underline{r}_c,\ \overline{r}_c]$, $\forall t \geq t_0$ holds if $\hat{r}_i(t_0) \in [\underline{r}_c,\ \overline{r}_c]$. Similarly, $\hat{\omega}_i(t) \in [\underline{\omega}_c,\ \overline{\omega}_c]$, $\forall t \geq t_0$ holds if $\hat{\omega}_i(t_0) \in [\underline{\omega}_c,\ \overline{\omega}_c]$. By assumption [A2], k_ω satisfying (33) exists. It follows from (19) and (33) that $\omega_i(t) \in [-\omega_{\max},\ \omega_{\max}]$, $\forall t \geq t_0$. Since k_ω meets (33), k_v satisfying (34) exists. It follows from (25) and (33) that $v_i(t) \in [v_{\min},\ v_{\max}]$, $\forall t \geq t_0$. The proof is thus completed. ■

Remark 4.2 In Marshall et al. (2004), Ceccarelli et al. (2008) and Sepulchre et al. (2007, 2008), vehicles were assumed to have identical constant v_i and a circular formation can be achieved by controlling ω_i. As pointed out in Seyboth et al. (2014), the assumption of identical v_i may not be satisfied in practice. Note that the initial state $[\hat{q}_i^{i\mathrm{T}}(t_0)\ \hat{r}_i(t_0)\ \hat{\omega}_i(t_0)]^{\mathrm{T}}$ can be selected based on the initial velocities $v_i(t_0)$ and $\omega_i(t_0)$. Our approach can address the case where vehicles initially have nonidentical v_i and do not know a common value $\omega_c r_c$. For this case, Seyboth et al. (2014) made vehicles converge to even spacing along a common circle while maintaining nonidentical constant v_i. This objective is not the same as that of this paper, since in Seyboth et al. (2014) the center of the circle is not prespecified and consensus of v_i (objective described in (4)) is not required. ■

Remark 4.3 Sepulchre et al. (2007) and Summers et al. (2009) required all vehicles to know the center of the common circle. Besides, in Sepulchre et al. (2007, 2008) and Summers et al. (2009), all vehicles need to know r_c and ω_c. By contrast, we only require one anonymous vehicle to know such information. One advantage of our approach is that if q_c, r_c, and ω_c are required to change, we only need to command one vehicle and then all vehicles will converge to a new circular formation. Moreover, to achieve the convergence to a given circle, it is required in Summers et al. (2009) that either the initial heading angle of each vehicle is in a certain range, or the initial distance to the center is larger than the desired radius. Furthermore, in Sepulchre et al. (2007, 2008), a common reference direction was required for obtaining the relative positions and the total number of vehicles N had to be known to each vehicle. While with our method, vehicles with any initial heading angles and any initial distances to the center can converge to a given circle. Each vehicle can only use its local coordinate frame and does not require the knowledge of N. ■

Remark 4.4 Seyboth et al. (2014) and Sepulchre et al. (2007, 2008) also considered the problem of making all orientations synchronized, i.e., $\theta_i = \theta_j, \forall i, j$. To this end, we can simply modify the definition of ϕ_i in (15) as $\phi_i = \theta^i_{i+} = \theta_{i+} - \theta_i, \forall i$. In this case, $\sum_{i=1}^N \phi_i \equiv 0$ and it follows from the proof of Lemma 3.2 that $\lim_{t\to\infty}(\theta_{i+}(t) - \theta_i(t)) = 0,\ \forall i$. It is noted that with our method, the switching between even spacing and orientation synchronization only requires to switch the controller of vehicle l, while in Sepulchre et al. (2007, 2008) and Seyboth et al. (2014), all vehicles need to switch the sign of gain K in their controllers. Based on this extension, we may further set $\omega_c = 0$ and $r_c > 0$, and let (25) switch to $v_i = k_v \hat{r}_i$. Then, all vehicles will switch to the parallel motion towards the same orientation as in Sepulchre et al. (2008). ■

5 An Illustrative Example

Consider a group of 5 nonholonomic vehicles (1) with labels 1-5 in the communication digraph $\mathcal{G}$ (see Fig. 1(b)). By default, all variables are in SI units.

Set $l = 1$ and the velocity constraint (32) is given as $v_i \in [3 - 1.8\sqrt{2},\ 3 + 1.8\sqrt{2}]$ and $\omega_i \in$ [-0.4, 0.4]. The bounds in assumption [A2] are given as $\underline{r}_c = 8$, $\overline{r}_c = 12$, $\underline{\omega}_c = 0.15$, and $\overline{\omega}_c = 0.3$. Based on Proposition 4.1, the design parameters in (19) and (25) are tuned as $\mu = 2,\ \nu = 0.5,\ k_\omega = 0.0186$, and $k_v = 0.5965$.

In this example, the formation parameter is initially given by $q_c = [0\ 0]^{\mathrm{T}},\ r_c = 8,\ \omega_c = 0.3$ and is switched to $q_c = [\text{-}40\ 20]^{\mathrm{T}},\ r_c = 10,\ \omega_c = 0.2$ at $t_s = 160$s. Besides, vehicle 5 will leave the group at the time instant t_s. From t_s onwards, vehicle 4 will be the *pre-neighbor* of vehicle 1. The initial states of vehicles are given as $\theta_1(0) = \pi$, $\theta_2(0) = \text{-}5\pi/6$, $\theta_3(0) = 0$, $\theta_4(0) = \text{-}\pi/3$, $\theta_5(0) = \pi/6$, $p_1(0) = [\text{-}30\ 10]^{\mathrm{T}}$, $p_2(0) = [\text{-}20\ \text{-}20]^{\mathrm{T}}$, $p_3(0) = [0\ \text{-}30]^{\mathrm{T}}$, $p_4(0) = [\text{-}20\ 10]^{\mathrm{T}}$, and $p_5(0) = [20\ 10]^{\mathrm{T}}$. The initial states $[\hat{q}_i^{i\mathrm{T}}(0)\ \hat{r}_i(0)\ \hat{\omega}_i(0)]^{\mathrm{T}}$ of observer (16)-(18) are randomly chosen in $\mathbb{R}^2 \times [\underline{r}_c,\ \bar{r}_c] \times [\underline{\omega}_c,\ \overline{\omega}_c]$.

Fig. 2 shows the trajectories of all vehicles. In Fig. 2, at first, vehicles converge to the circular formation with $q_c = [0\ 0]^{\mathrm{T}}$ and $r_c = 8$. When q_c and r_c are switched and vehicle 5 leaves, the remaining 4 vehicles resume even spacing and converge to the new circular formation with $q_c = [\text{-}40\ 20]^{\mathrm{T}}$ and $r_c = 10$. Fig. 3 shows that the relative distance between each vehicle and the center converges to the given radius. Fig. 4 illustrates that the convergence to even spacing can be achieved independent of the total number of vehicles N. Fig. 5 and Fig. 6 not only show that all velocities converge to the given values, but also indicate that velocity constraint (32) is always satisfied. These simulation results verify effectiveness of the proposed control law.

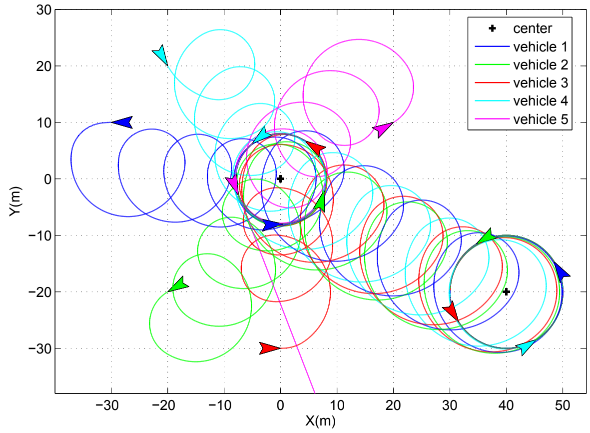


Fig. 2. Trajectories of vehicles p_i.

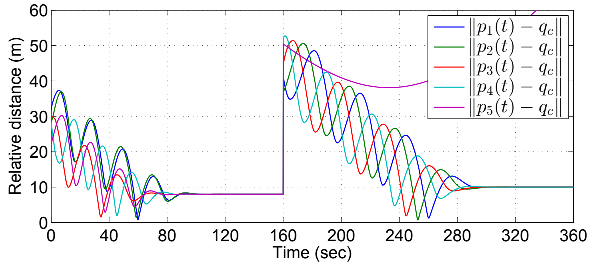


Fig. 3. Relative distance $\|p_i - q_c\|$.

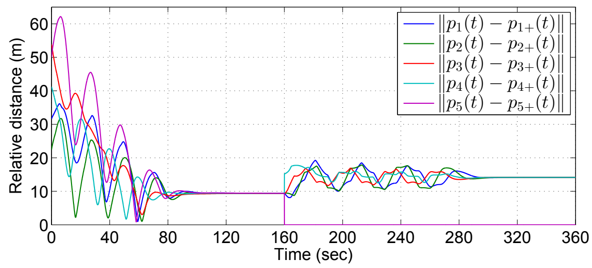


Fig. 4. Relative distance $\|p_i - p_{i+}\|$.

6 Conclusion

In this paper, we have proposed a distributed circular formation control law for ring-networked nonholonomic vehicles with local coordinate frames. Only one anonymous vehicle is required to know the parameter describing the prespecified circular formation. Furthermore, the velocity constraint including positive-minimum linear velocity is considered. In the future, we will investigate the circular formation with time-varying formation parameter, unknown disturbance and collision avoidance.

A Proof of Lemma 3.1

Define $\tilde{q}_i = \hat{q}^i_i - q^i_c$, $\tilde{r}_i = \hat{r}_i - r_c$, and $\tilde{\omega}_i = \hat{\omega}_i - \omega_c$. Denote $\tilde{r} = \mathrm{col}(\tilde{r}_l, \tilde{r}_{l+}, ..., \tilde{r}_{l-})$ and $\tilde{\omega} = \mathrm{col}(\tilde{\omega}_l, \tilde{\omega}_{l+}, ..., \tilde{\omega}_{l-})$.

First, using (17)-(18), we have

$$\dot{\tilde{r}} = -H\tilde{r},\ \dot{\tilde{\omega}} = -H\tilde{\omega}, \tag{A.1}$$

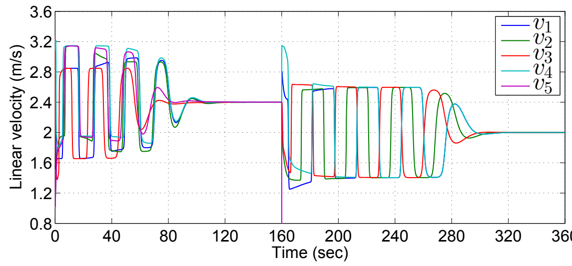


Fig. 5. Linear velocity v_i.

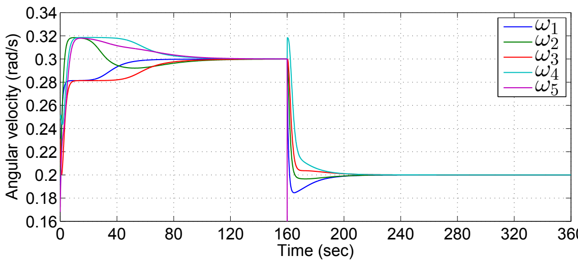


Fig. 6. Angular velocity ω_i.

where $H = \{h_{kj}\}$ is a lower-triangular matrix with $h_{kk} = 1$, $k = 1, ..., N$; $h_{k(k\text{-}1)} = \text{-}1$, $k = 2, ..., N$, and others all zero. Thus, $-H$ is Hurwitz and system (A.1) is globally exponentially stable.

Next, since $q_c^i = p_{i\text{-}}^i + R(\phi_{i\text{-}})q_c^{i\text{-}}$ and $p_{i\text{-}}^i = -R(\phi_{i\text{-}})p_i^{i\text{-}}$,

$$-R(\phi_{i\text{-}})p_i^{i\text{-}} = q_c^i - R(\phi_{i\text{-}})q_c^{i\text{-}}. \tag{A.2}$$

Using (A.2), it follows from (13) and (16) that

$$\begin{aligned}
\dot{\tilde{q}}_l &= \omega_l S\tilde{q}_l - \tilde{q}_l, && \text{(A.3a)}\\
\dot{\tilde{q}}_{l+} &= \omega_{l+} S\tilde{q}_{l+} - \tilde{q}_{l+} + R(\phi_l)\tilde{q}_l, && \text{(A.3b)}\\
&\vdots\\
\dot{\tilde{q}}_{l\text{-}} &= \omega_{l\text{-}} S\tilde{q}_{l\text{-}} - \tilde{q}_{l\text{-}} + R(\phi_{l\text{--}})\tilde{q}_{l\text{--}}. && \text{(A.3c)}
\end{aligned}$$

Consider system (A.3a) and a Lyapunov function candidate $V_l(\tilde{q}_l) = \frac{1}{2}\tilde{q}_l^{\mathrm{T}}\tilde{q}_l$. Taking the time derivative of $V_l(\tilde{q}_l)$ along the trajectories of system (A.3a) yields

$$\dot{V}_l(\tilde{q}_l) = \tfrac{1}{2}\omega_l\tilde{q}_l^{\mathrm{T}}(S + S^{\mathrm{T}})\tilde{q}_l - \tilde{q}_l^{\mathrm{T}}\tilde{q}_l \leq -\|\tilde{q}_l\|^2 \leq 0. \tag{A.4}$$

Since $V_l(\tilde{q}_l) = \frac{1}{2}\|\tilde{q}_l\|^2$, it follows from (A.4) that $d\|\tilde{q}_l\|/dt \leq -\|\tilde{q}_l\|$ when $\|\tilde{q}_l\| \neq 0$, and $\mathrm{D}^+\|\tilde{q}_l\| \leq 0$ when $\|\tilde{q}_l\| = 0$. Thus, $\mathrm{D}^+\|\tilde{q}_l\| \leq -\|\tilde{q}_l\|$ holds for all $\|\tilde{q}_l\|$. By the comparison lemma, we have

$$\|\tilde{q}_l(t)\| \leq e^{\text{-}(t\text{-}t_0)}\|\tilde{q}_l(t_0)\|. \tag{A.5}$$

Then, consider system (A.3b) and a Lyapunov function candidate $V_{l+}(\tilde{q}_{l+}) = \frac{1}{2}\tilde{q}_{l+}^{\mathrm{T}}\tilde{q}_{l+}$. Taking its time derivative along the trajectories of system (A.3b) gives

$$\begin{aligned}
\dot{V}_{l+}(\tilde{q}_{l+}) &= \tfrac{1}{2}(\tilde{q}_{l+}^{\mathrm{T}}R(\phi_l)\tilde{q}_l + \tilde{q}_l^{\mathrm{T}}R^{\mathrm{T}}(\phi_l)\tilde{q}_{l+}) - \tilde{q}_{l+}^{\mathrm{T}}\tilde{q}_{l+}\\
&\leq \|\tilde{q}_{l+}\|\|R(\phi_l)\tilde{q}_l\| - \|\tilde{q}_{l+}\|^2 = \|\tilde{q}_{l+}\|\|\tilde{q}_l\| - \|\tilde{q}_{l+}\|^2,
\end{aligned} \tag{A.6}$$

where it is noted that $\|R(\phi_l)\tilde{q}_l\| = \|\tilde{q}_l\|$. Similarly, it follows from (A.6) that $\mathrm{D}^+\|\tilde{q}_{l+}\| \leq \|\tilde{q}_l\| - \|\tilde{q}_{l+}\|$ for all $\|\tilde{q}_{l+}\|$. By the comparison lemma and (A.5), $\|\tilde{q}_{l+}(t)\|$ satisfies

$$\begin{aligned}
\|\tilde{q}_{l+}(t)\| &\leq e^{\text{-}(t\text{-}t_0)}\|\tilde{q}_{l+}(t_0)\| + \textstyle\int_{t_0}^t e^{\text{-}(t\text{-}\tau)}\|\tilde{q}_l(\tau)\|d\tau\\
&\leq e^{\text{-}(t\text{-}t_0)}\|\tilde{q}_{l+}(t_0)\| + \|\tilde{q}_l(t_0)\|\textstyle\int_{t_0}^t e^{\text{-}(t\text{-}\tau)}e^{\text{-}(\tau\text{-}t_0)}d\tau\\
&\leq e^{\text{-}(t\text{-}t_0)}\|\tilde{q}_{l+}(t_0)\| + e^{\text{-}(t\text{-}t_0)}(t - t_0)\|\tilde{q}_l(t_0)\|.
\end{aligned}$$

Thus, letting $l = 1$ and using induction, we can obtain

$$\|\tilde{q}_i(t)\| \leq e^{\text{-}(t\text{-}t_0)}\textstyle\sum_{k=1}^i \frac{(t-t_0)^{k\text{-}1}}{(k-1)!}\|\tilde{q}_{i\text{-}k+1}(t_0)\|, \forall i \in [1, N],$$

which concludes that all $\|\tilde{q}_i(t)\|$ converge to 0 exponentially as $t \to \infty$. The proof is thus completed. ■

References

Ceccarelli, N., Di Marco, M., Garulli, A., & Giannitrapani, A. (2008). Collective circular motion of multi-vehicle systems. *Automatica*, *44*(12), 3025–3035.

Chen, Z., & Zhang, H.-T. (2011). No-beacon collective circular motion of jointly connected multi-agents. *Automatica*, *47*(9), 1929–1937.

Chen, Z., & Zhang, H.-T. (2013). A remark on collective circular motion of heterogeneous multi-agents. *Automatica*, *49*(5), 1236–1241.

Dixon, W. E., Dawson, D. M., Zergeroglu, E., & Behal, A. (2001). *Nonlinear control of wheeled mobile robots*. Springer-Verlag New York, Inc.

El-Hawwary, M. I., & Maggiore, M. (2013). Distributed circular formation stabilization for dynamic unicycles. *IEEE Transactions on Automatic Control*, *58*(1), 149–162.

Lan, Y., Yan, G., & Lin, Z. (2010). Distributed control of cooperative target enclosing based on reachability and invariance analysis. *Systems & Control Letters*, *59*(7), 381–389.

Leonard, N. E., Paley, D. A., Lekien, F., Sepulchre, R., Fratantoni, D. M., & Davis, R. E. (2007). Collective motion, sensor networks, and ocean sampling. *Proceedings of the IEEE*, *95*(1), 48–74.

Lin, Y., Sontag, E. D., & Wang, Y. (1996). A smooth converse Lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, *34*(1), 124–160.

Marshall, J. A., Broucke, M. E., & Francis, B. A. (2004). Formations of vehicles in cyclic pursuit. *IEEE Transactions on Automatic Control*, *49*(11), 1963–1974.

Marshall, J. A., Broucke, M. E., & Francis, B. A. (2006a). Pursuit formations of unicycles. *Automatica*, *42*(1), 3–12.

Marshall, J. A., Fung, T., Broucke, M. E., D'Eleuterio, G. M., & Francis, B. A. (2006b). Experiments in multirobot coordination. *Robotics and Autonomous Systems*, *54*(3), 265–275.

Pavone, M., & Frazzoli, E. (2007). Decentralized policies for geometric pattern formation and path coverage. *Journal of Dynamic Systems, Measurement, and Control*, *129*(5), 633–643.

Ramirez-Riberos, J. L., Pavone, M., Frazzoli, E., & Miller, D. W. (2010). Distributed control of spacecraft formations via cyclic pursuit: Theory and experiments. *Journal of Guidance, Control, and Dynamics*, *33*(5), 1655–1669.

Ren, W. (2008). On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, *53*(6), 1503–1509.

Ren, W. (2009). Collective motion from consensus with cartesian coordinate coupling. *IEEE Transactions on Automatic Control*, *54*(6), 1330–1335.

Ren, W., & Atkins, E. (2007). Distributed multi-vehicle coordinated control via local information exchange. *International Journal of Robust and Nonlinear Control*, *17*(10-11), 1002–1033.

Ren, W., & Beard, R. W. (2004). Trajectory tracking for unmanned air vehicles with velocity and heading rate constraints. *IEEE Transactions on Control Systems Technology*, *12*(5), 706–716.

Sepulchre, R., Paley, D. A., & Leonard, N. E. (2007). Stabilization of planar collective motion: All-to-all communication. *IEEE Transactions on Automatic Control*, *52*(5), 811–824.

Sepulchre, R., Paley, D. A., & Leonard, N. E. (2008). Stabilization of planar collective motion with limited communication. *IEEE Transactions on Automatic Control*, *53*(3), 706–719.

Seyboth, G. S., Wu, J., Qin, J., Yu, C., & Allgower, F. (2014). Collective circular motion of unicycle type vehicles with nonidentical constant velocities. *IEEE Transactions on Control of Network Systems*, *1*(2), 167–176.

Sharma, B. R., Ramakrishnan, S., & Kumar, M. (2013). Cyclic pursuit in a multi-agent robotic system with double-integrator dynamics under linear interactions. *Robotica*, *31*(07), 1037–1050.

Sinha, A., & Ghose, D. (2007). Generalization of nonlinear cyclic pursuit. *Automatica*, *43*(11), 1954–1960.

Summers, T. H., Akella, M. R., & Mears, M. J. (2009). Coordinated standoff tracking of moving targets: control laws and information architectures. *Journal of Guidance, Control, and Dynamics*, *32*(1), 56–69.

Yu, X., & Liu, L. (2016). Distributed formation control of nonholonomic vehicles subject to velocity constraints. *IEEE Transactions on Industrial Electronics*, *63*(10), 1289–1298.

Zheng, R., Lin, Z., Fu, M., & Sun, D. (2015). Distributed control for uniform circumnavigation of ring-coupled unicycles. *Automatica*, *53*, 23–29.

Zheng, R., Lin, Z., & Yan, G. (2009). Ring-coupled unicycles: boundedness, convergence, and control. *Automatica*, *45*(11), 2699–2706.